Life Insurance
– Lecture Parts III and IV –

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Course Information

- Lecturers:
  - Feiko Drost (I: micro longevity risk and II: interest rate risk)
  - Christoph Hambel (III: macro longevity risk and IV: all risks combined)
  - Okke Labout (tutorials)

- The second half of this course ...
  - ... provides an introduction to macro longevity risk and to applications in actuarial science that combine all types of risk.
  - ... directly builds upon the first half and does not require any additional pre-knowledge.

- Grading:
  - Exam 70%
  - Two Assignments (15% each)

What to expect?

- What can you expect from me? I will...
  - ... timely provide the learning material on Canvas
  - ... also upload the slides with hand-written complements (some parts of the slides are intentionally blank)
  - ... illustrate the lecture by examples
  - ... provide a lot of problems to practice the material
  - ... be available for questions
- What will I expect from you? You should ...
  - ... be well-prepared when you come to the lecture
  - ... actively participate in the lecture
  - ... take the opportunity and ask me questions during the classes

Structure of the Course (Second Half)

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Part III: Macro Longevity Risk

1. Introduction
2. Relevance of Macro Longevity Risk
   - First Pillar: AOW
   - Second Pillar: Pension Funds
3. Modeling Mortality
4. Benchmark Model
   - The Lee-Carter Model
   - Alternative Estimation
   - Some Applications and Extensions
5. The AG2022 Model and COVID-19
   - Model and Projections
   - Closure of the Life Table
6. Model Risk: A Very Brief Introduction

Part IV: Pricing under all Types of Risk

7. Setting

8. Illustrations
    - No risk
    - Micro longevity risk
    - Macro longevity risk
    - Interest rate risk
    - All risks combined

Course Learning Goals

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After successful completion of the course, the student will be able to ...

- CG1 calculate the prices of basic term life insurance and annuity products.
- CG2 determine prices of fixed-income securities with term structure models.
- CG3 apply mathematical and statistical tools to estimate and solve macro longevity risk models.
- CG4 compute software-based simulations, statistics, and visualizations of micro- and macro longevity risk as well as interest rate risk models based on real-world data.
- CG5 analyze the effect of pandemics on macro longevity risk using the model of the Royal Dutch Actuarial Association.

Part III

Macro Longevity Risk

Table of Contents

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1. Introduction
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Recall: Longevity Risk

$T^{(g)}_{x,t}$: time of death is a RV ; $p^{(g)}_{x,t} = \mathbb{P}(T^{(g)}_{x,t} \geq 1)$

$q^{(g)}_{x,t} = 1 - p^{(g)}_{x,t}$

- **Micro Longevity Risk**

  Risk because (for given death probabilities) an individual's *remaining lifetime* is unknown.

  The remaining lifetime of an individual of age x belonging to a group g at time t is modeled as a random variable conditional on the future death probabilities $q^{(g)}_{x+s,t+s}$, $s = 0, 1, 2, \ldots$

- **Macro Longevity Risk**

  Additional risk because future death probabilities are unknown.

  The future death probabilities $q^{(g)}_{x+s,t+s}$, $s = 0, 1, 2, \ldots$, will be modeled as random variables on date t.

Macro longevity risk: $q^{(g)}_{x,t}$ are RVs

The life table for a given group g can be represented as

$\rightarrow$ time

$\downarrow$ age

$$\begin{pmatrix} q^{(g)}_{0,t_0} & q^{(g)}_{0,t_0+1} & \cdots & q^{(g)}_{0,t_0+T} & \cdots \\ q^{(g)}_{1,t_0} & q^{(g)}_{1,t_0+1} & \cdots & q^{(g)}_{1,t_0+T} & \cdots \\ q^{(g)}_{2,t_0} & q^{(g)}_{2,t_0+1} & \cdots & q^{(g)}_{2,t_0+T} & \cdots \\ \vdots & & & \vdots & \\ \vdots & & & \vdots & \\ q^{(g)}_{x_{\max},t_0} & q^{(g)}_{x_{\max},t_0+1} & \cdots & q^{(g)}_{x_{\max},t_0+T} & \cdots \end{pmatrix} \begin{matrix} \rightarrow \textit{Predictions} \\ \rightarrow \textit{Predictions} \\ \rightarrow \textit{Predictions} \\ \\ \\ \rightarrow \textit{Predictions} \end{matrix}$$

AG2022: $x_{\max=120}$, observed: $t_0 = 1970$, $t_0 + T = 2021$, predicted: $t_0 + T + s \geq 2022$

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1. How to estimate/calibrate the observed part? → *historical data*
2. How to determine the predictions and the uncertainty surrounding these predictions (macro longevity risk)? → *use a model to simulate those probabilities into the future: 1) Lee-Carter, 2) AG 2022/2024*

1970 *extrapolation*

$$
\begin{array}{c} \text{time} \rightarrow \\ \end{array}
$$

$$
\downarrow \text{age} \quad
\begin{pmatrix}
\hat{q}^{(g)}_{0,t_0} & \hat{q}^{(g)}_{0,t_0+1} & \cdots & \hat{q}^{(g)}_{0,t_0+T} & \cdots \\
\hat{q}^{(g)}_{1,t_0} & \hat{q}^{(g)}_{1,t_0+1} & \cdots & \hat{q}^{(g)}_{1,t_0+T} & \cdots \\
\hat{q}^{(g)}_{2,t_0} & \hat{q}^{(g)}_{2,t_0+1} & \cdots & \hat{q}^{(g)}_{2,t_0+T} & \cdots \\
\vdots & & & \vdots & \\
\vdots & & & \vdots & \\
\hat{q}^{(g)}_{x_{\max},t_0} & \hat{q}^{(g)}_{x_{\max},t_0+1} & \cdots & \hat{q}^{(g)}_{x_{\max},t_0+T} & \cdots
\end{pmatrix}
\begin{array}{l}
\rightarrow \textit{Predictions} \\
\rightarrow \textit{Predictions} \\
\rightarrow \textit{Predictions} \\
\\
\\
\rightarrow \textit{Predictions}
\end{array}
$$

Period Calculations and Cohort Calculations

- **Period Calculations** t → t+1, t+2, t+3, ...
  - Period calculations is using the columns (e.g., copy the final column) of a life table to predict the next period death probability.
  - This means that any future changes to mortality rates would not be taken into account.
  - Period life expectancies use mortality rates from a single year and assume that those rates apply throughout the remainder of a person's life.

- **Cohort Calulations**
  - Cohort calculations is taking future trends into account using models.
  - A cohort life table uses a combination of observed mortality rates for the cohort for past years and projections about mortality rates for the cohort for future years.
  - Requires a model.

→ Period life expectancy would match cohort life expectancy only if there were no changes in age-specific mortality rates over time.

Traditional Approach: Naive Forecast

Period Calculations = Naive Forecast

- Traditionally, macro longevity risk was ignored.
- One assumed that the most recently estimated period death probabilities hold true for all future years, i.e., for the cohort $(x, t_0 + T)$ one assumed

$$\underbrace{q^{(g)}_{x+s,t_0+T+s}}_{predicted} = \underbrace{q^{(g)}_{x+s,t_0+T}}_{observed}$$

  for all $s \geq 0$ and all ages x.
- This means that – if we ignore macro longevity risk – the entries in the last column of the observation part of the life table equal the entries of the prediction part.

Traditional Approach: Naive Forecast

$t = t_0 + T$

$$\hat{q}^{(g)}_{x+s,\,t+s} = \hat{q}^{(g)}_{x+s,\,t}$$

$\underline{s=0}$:

$$\hat{q}^{(g)}_{x,t} = \hat{q}^{(g)}_{x,t}$$

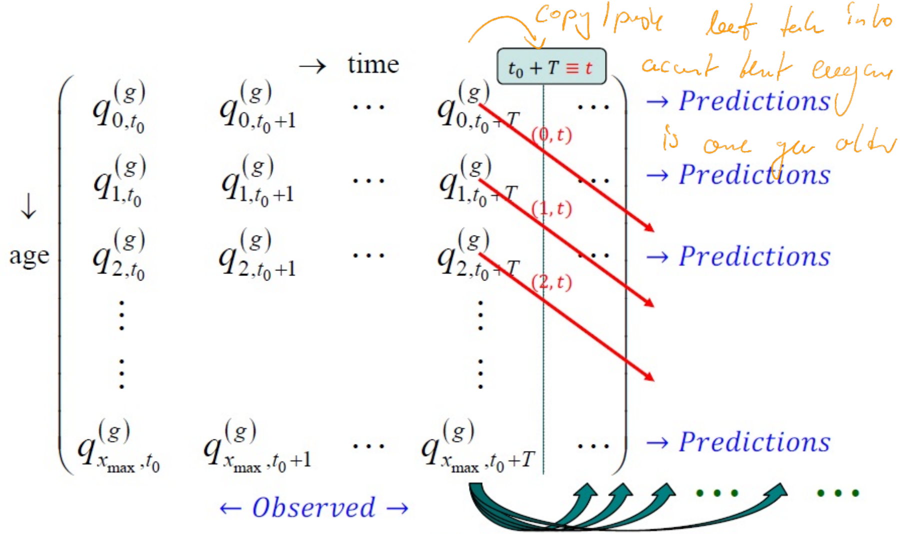
tautological for the current year $t = t_0 + T$

$\underline{s=1}$:

$$\underbrace{\hat{q}^{(g)}_{x+1,\,t+1}} = \hat{q}^{(g)}_{x+1,\,t}$$

prediction of 1 year death probability for the next year

Traditional Approach: Naive Forecast



$$q^{(g)}_{x+s,t_0+T+s} = q^{(g)}_{x+s,t_0+T}$$

Period vs. Cohort Calculations

Both period and cohort calculations have some drawbacks:

- **Drawbacks of period calculations**
  - Ignoring trends in death probabilities may lead to significant overestimation of death probabilities.
  - Ignoring uncertainty in future death probabilities may lead to significant underestimation of the risk in life insurance portfolios.
  - Sensitive to (transitory) shocks, e.g., WW2, Spanish flu, COVID-19.
- **Drawbacks of cohort calculations**
  - We unavoidably introduce model risk if we use forecasts.

"All models are wrong, but some are useful."

— George Box (1976)

Current Practice

- Statistics Netherlands (CBS) and the Royal Dutch Actuarial Association (AG) produce point forecasts for future one-year death probabilities by age and gender.

  → Are available on the website of the AG.

- These point forecasts are referred to as *best-estimate death probabilities*.

- The AG-models also easily allow the quantification of (at least part of the) macro longevity risk.

- To mitigate the effect of model risk, these best-estimate death probability forecasts are revised annually (CBS, December), or bi-annually (AG, September in even years).

Recall: Some Formulas for Cohort (x, t)

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- **τ-years-from-now survival probability:**

$$ {}_{\tau}p_{x,t}^{(g)} = \prod_{k=0}^{\tau-1} p_{x+k,t+k}^{(g)}, \qquad p_{x,t}^{(g)} = 1 - q_{x,t}^{(g)} $$

$q_{x,t}^{(g)} = \mathbb{P}(T_{x,t}^{(g)} \leq 1)$

- **Remaining life expectancy:**

$$ e_{x,t}^{(g)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{(g)} + 0.5 $$

- **Value of immediate single life annuity:**

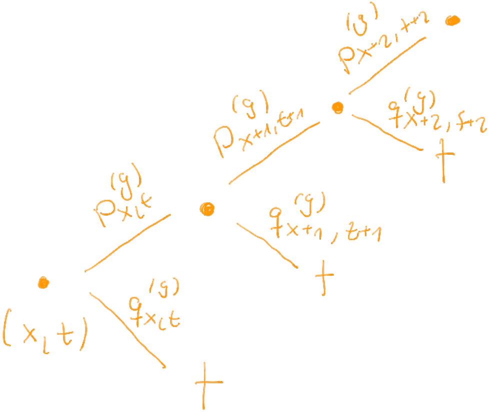
PV of $1, which you receive from now on every year until you die

$$ a_{x,t}^{(g)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{(g)} \frac{1}{\left(1 + R_t(t+\tau)\right)^{\tau}} $$

- **Value of T-years deferred single life annuity:**

$$ a_{x,t}^{(g)}(T) = \sum_{\tau=T}^{\infty} {}_{\tau}p_{x,t}^{(g)} \frac{1}{\left(1 + R_t(t+\tau)\right)^{\tau}} $$

Recall: Some Formulas for Cohort (x, t)

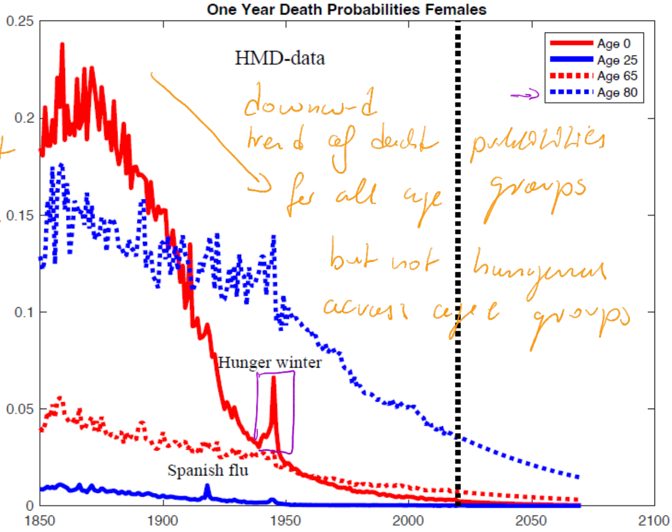


$$\mathbb{P}\left(T^{(g)}_{x,t} \geq 3\right)$$

$$= {}_3p^{(g)}_{x,t}$$

$$= \prod_{s=0}^{2} p^{(g)}_{x+s,\,t+s}$$

One-Year Death Probabilities

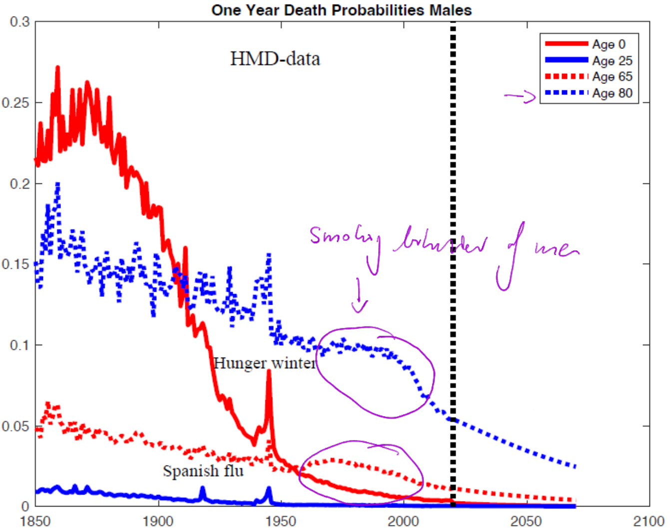


extreme shocks may affect different age groups differently

downward trend of death probabilities for all age groups

but not homogeneous across age groups

One-Year Death Probabilities



Remaining Life Expectancy (Newborns)

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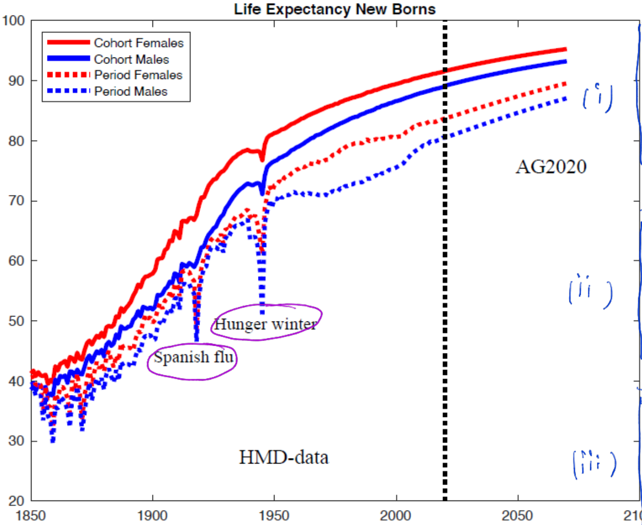

Cohort LE > Period LE

Men's LE < Wom's LE

Cohort LE is much less sensitive to extend shock than Period LE

Remaining Life Expectancy (Age 25)

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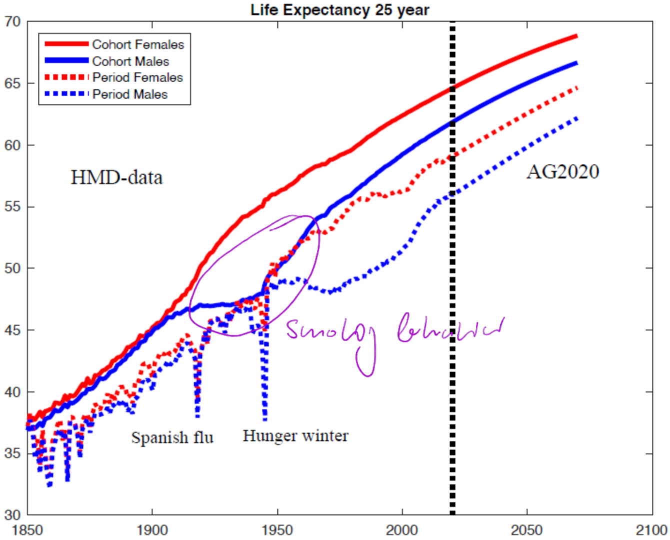

Remaining Life Expectancy (Age 65)

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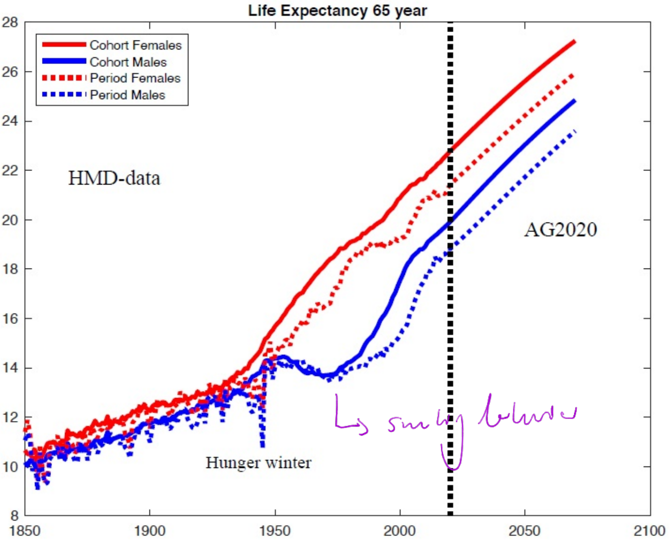

Remaining Life Expectancy (Age 80)

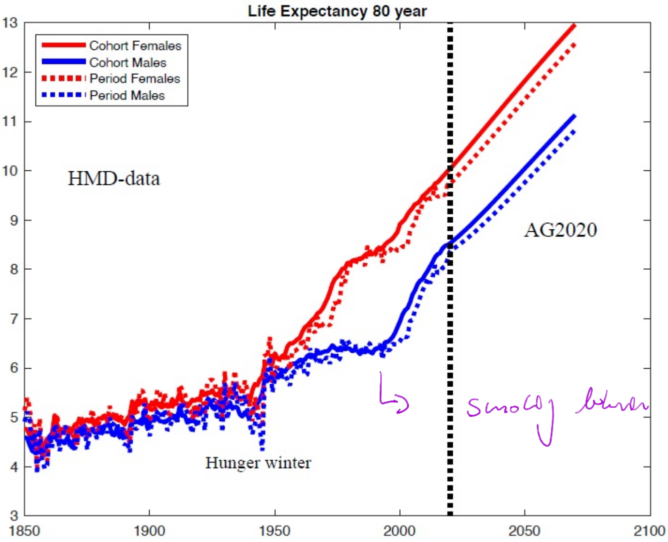

Table of Contents

1 Introduction

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- Second Pillar: Pension Funds

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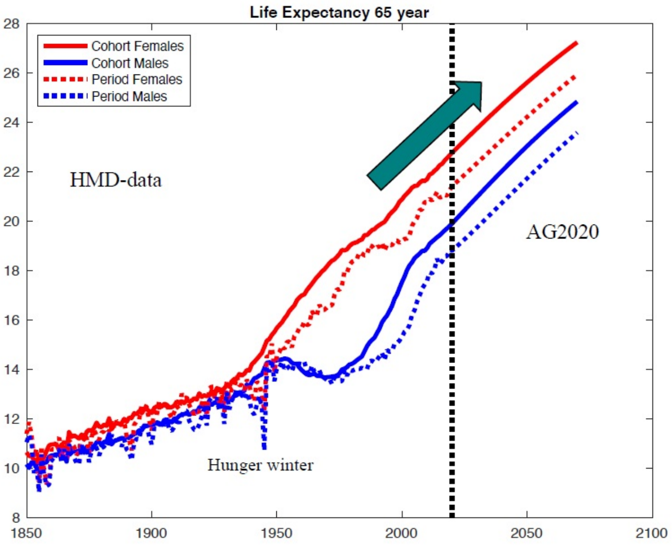
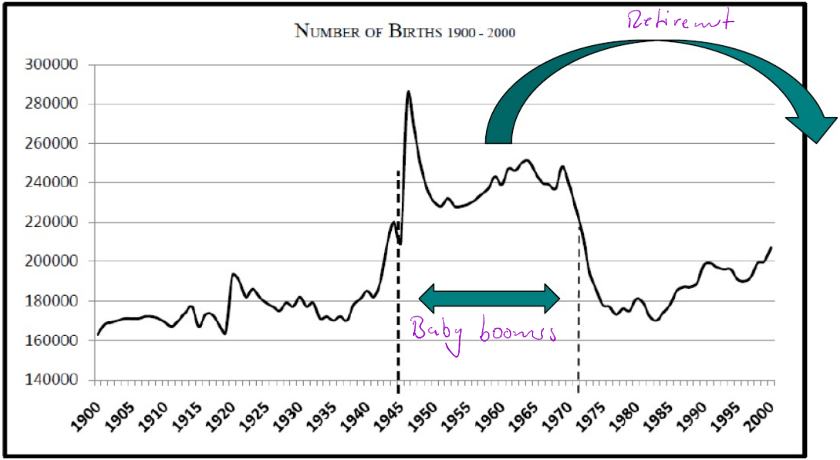
First Pillar: AOW

- AOW is the basic pension in the Netherlands that everyone gets, who lived in the Netherlands.
- This pillar is not related to how much the retiree worked.
- The pension depends on how many years the retiree lived in the Netherlands before retirement.
- If the retiree lived the fifty years before retirement in the Netherlands, he/she gets the full amount. If someone lived a shorter period of time in the Netherlands, this amount will be scaled down proportionally.
- Changes in life expectancy can affect whether the government can afford AOW.
  - Life expectancy has increased dramatically during the last decades.
  - It is unclear whether and how it will continue to increase (macro longevity risk).
- Other factors such as the number of newborns influence the stability and sustainability of the pension system.

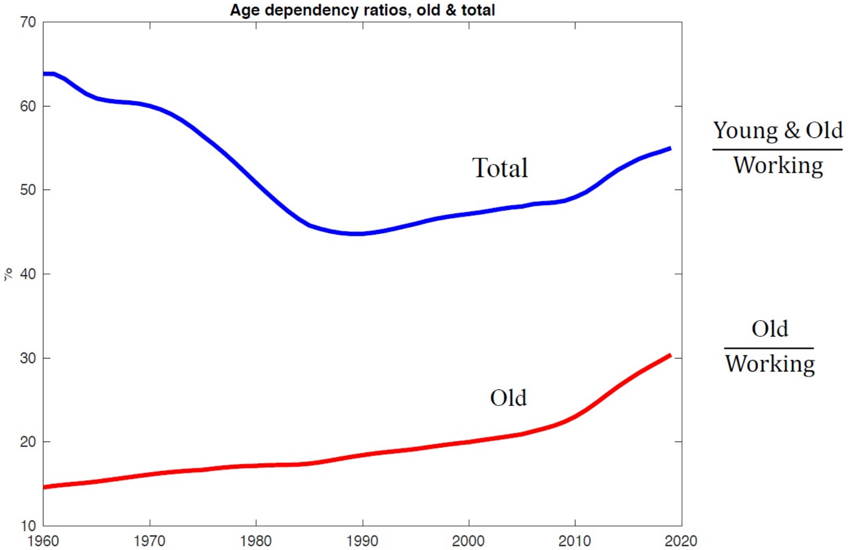
Increase in Life Expectancy

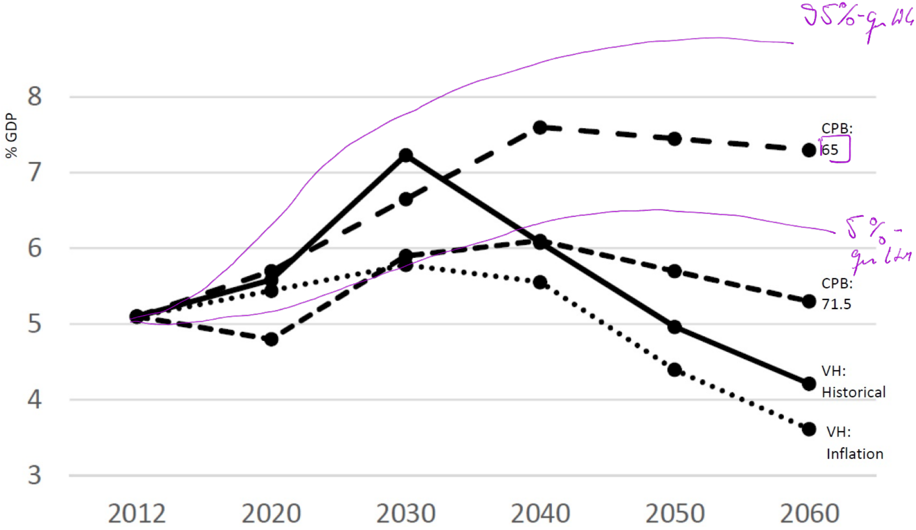


Number of Newborns (Source: CBS)

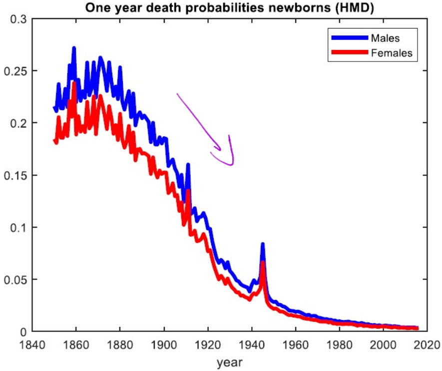


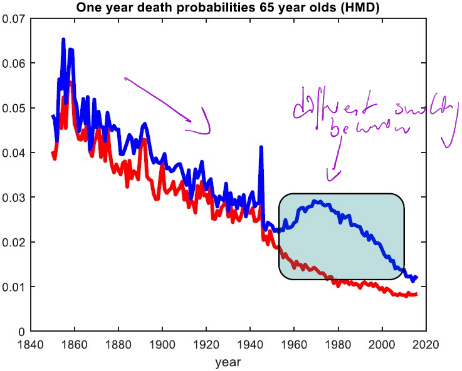
Dependency Ratio (Source: World Bank)



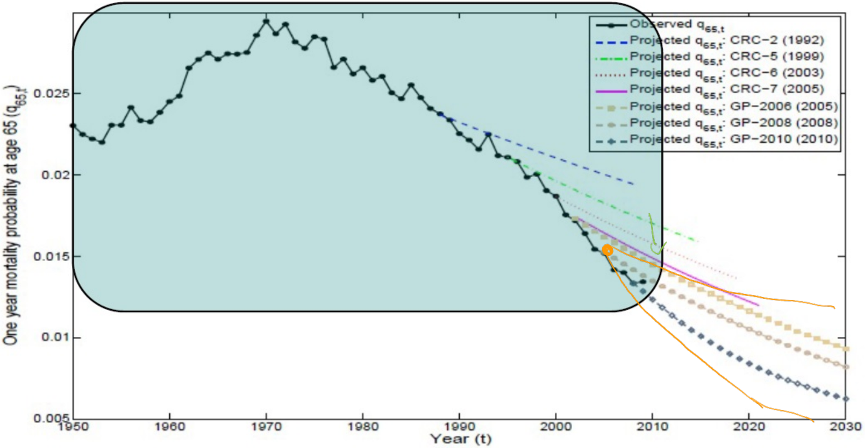

% GDP
8
7
6
5
4
3
2012
2020
2030
2040
2050
2060
CPB:
65
CPB:
71.5
VH:
Historical
VH:
Inflation

Possible Future Scenarios





Possible Future Scenarios



- Best estimate projections were wrong in the past!
- Macro Longevity Risk: Need to quantify the uncertainty around the projections as well.

Current Practice Revisited

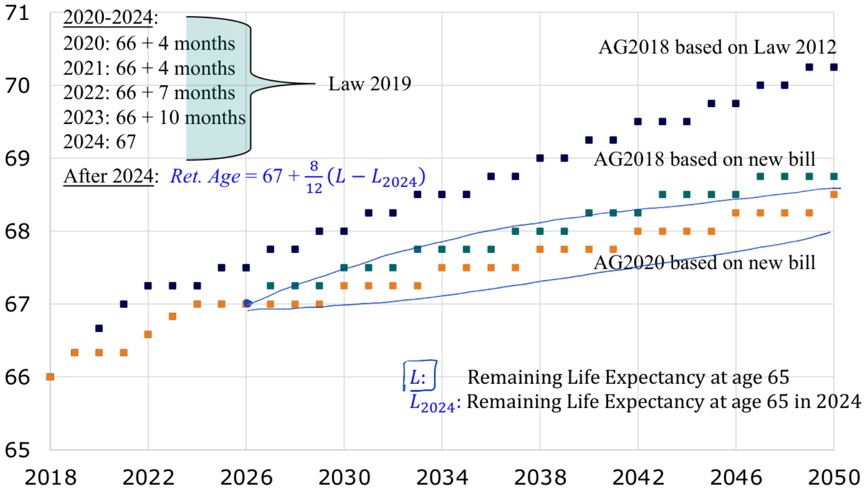
- Statistics Netherlands (CBS) and the Royal Dutch Actuarial Association produce point forecasts for future one-year death probabilities by age and gender.

  → Are available on the website of the AG.

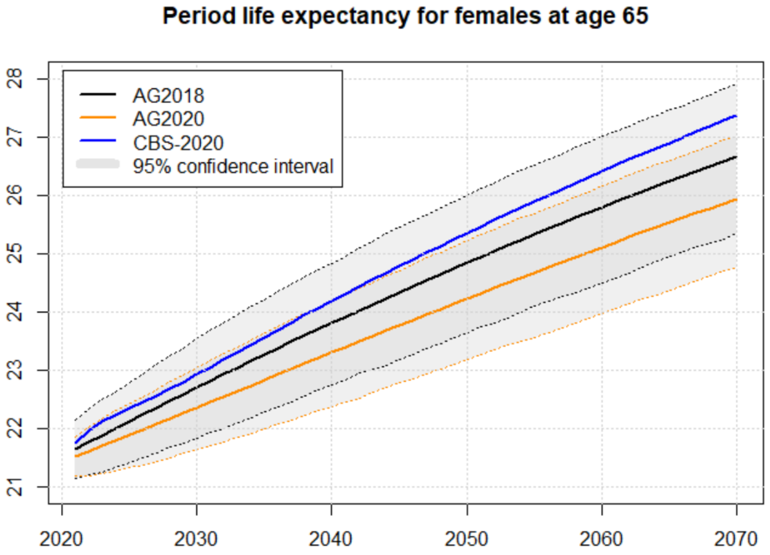
- These point forecasts ("best-estimate" death probabilities) are nowadays based on underlying models. These models can also be used to quantify macro longevity risk, for example, in terms of confidence intervals around the point forecasts.

- Part III of the course is going to illustrate this.
  - The models are not only used to derive the best estimates.
  - They can also be used to estimate confidence intervals describing the uncertainty around the point estimates.

Illustrating Macro Longevity Risk



Illustrating Macro Longevity Risk



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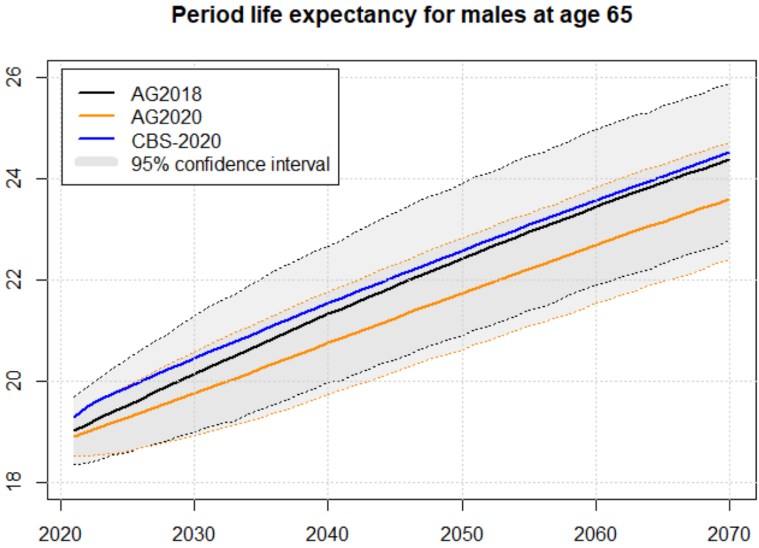
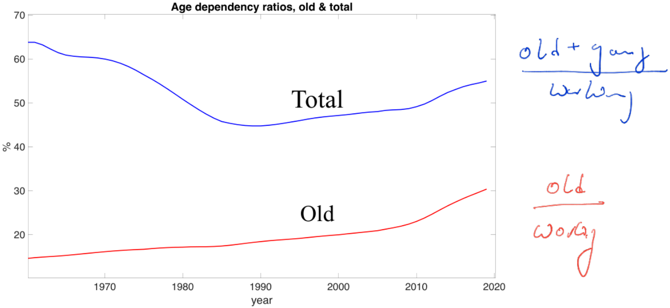
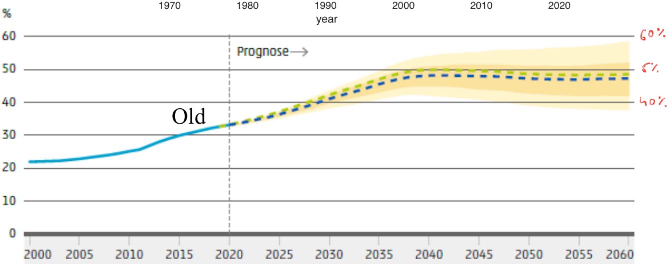


Illustrating Macro Longevity Risk





Pension Contract

- Describes the build-up of pension entitlements.
  - Many different contracts.
  - Alternative build-up percentages (around 2%).
  - Alternative ambitions (nominal, real).
  - Alternative indexation rules (price inflation, wage inflation).
- Changes all the time...
- New pension contract under construction...

- Building up entitlements (for some given year t):

$$\pi_x = \sum_{\tau = T_{Ret,x}}^{\infty} {}_{\tau - x}p_x \frac{f_x}{\left(1 + R(\tau - x)\right)^{\tau - x}}$$

- Notation:

x : age

π_x : pension contribution

f_x : pension entitlement per year

${}_{\tau - x}p_x$: $(\tau - x)$-years survival probability of an individual of age x

$R(\tau - x)$: discount rate with maturity $(\tau - x)$ years

$T_{Ret,x}$: retirement age for the generation of age x

Contributions, Entitlements, and Pensions

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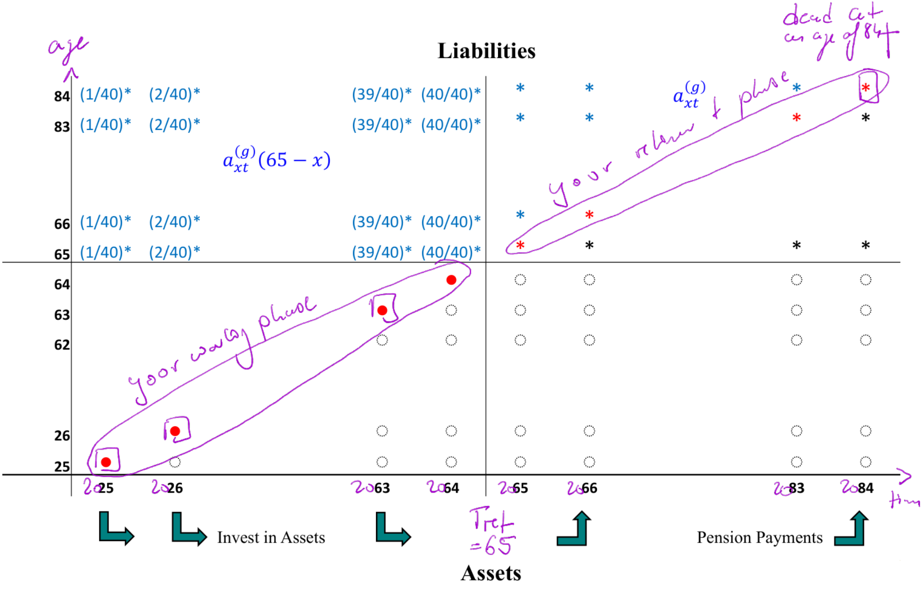

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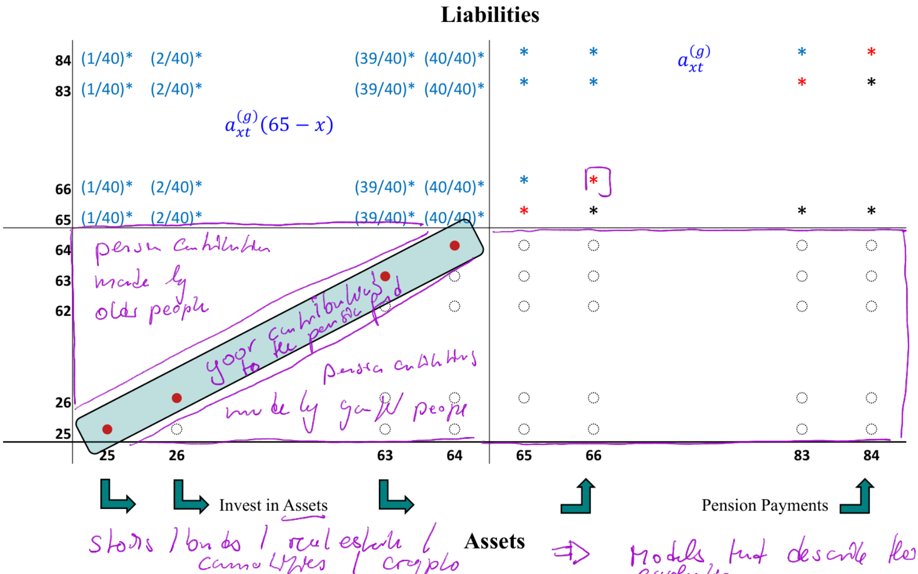
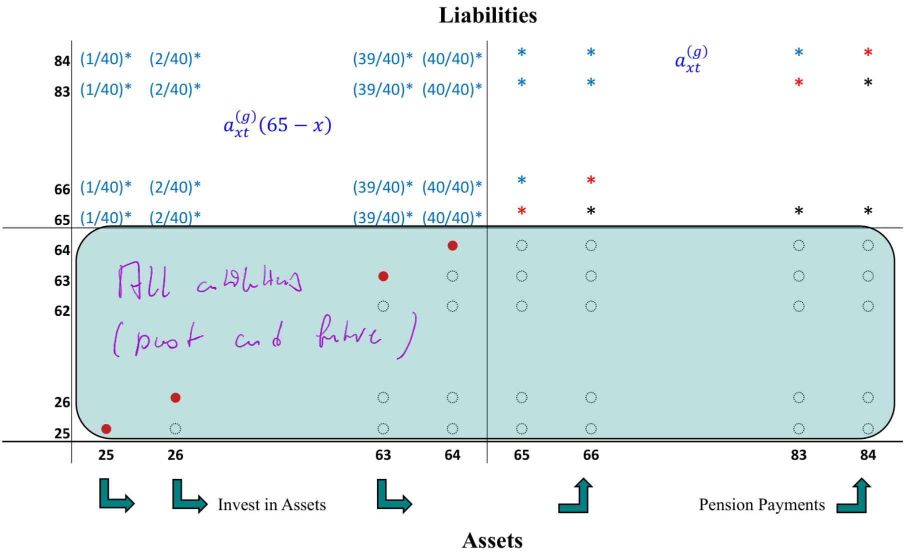
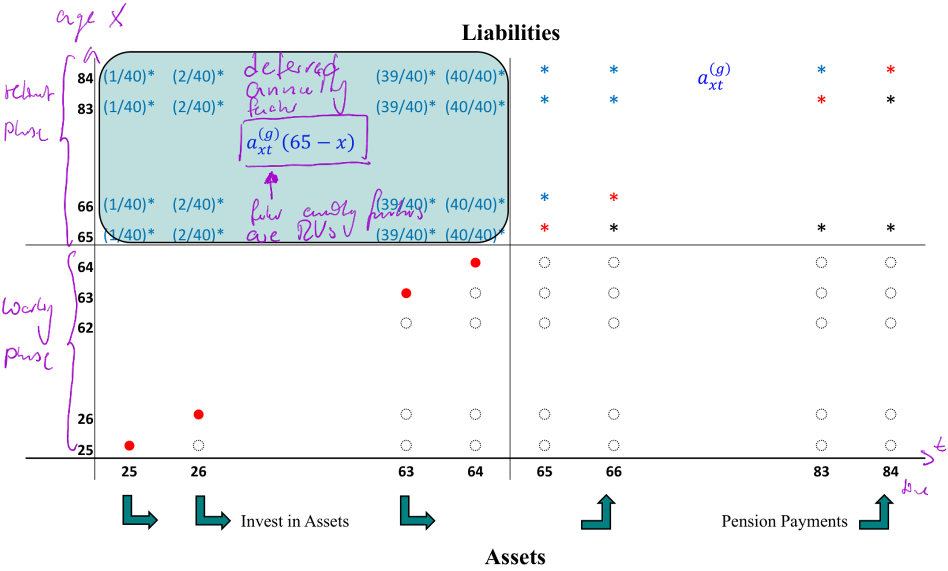



Liabilities
84 (1/40)* (2/40)* (39/40)* (40/40)*
83 (1/40)* (2/40)* (39/40)* (40/40)*
$a_{xt}^{(g)}(65-x)$
$a_{xt}^{(g)}$
66 (1/40)* (2/40)* (39/40)* (40/40)*
65 (1/40)* (2/40)* (39/40)* (40/40)*
64
63
62
26
25
All liabilities
(past and future)
25 26 63 64 65 66 83 84
Invest in Assets
Pension Payments
Assets

Contributions, Entitlements, and Pensions



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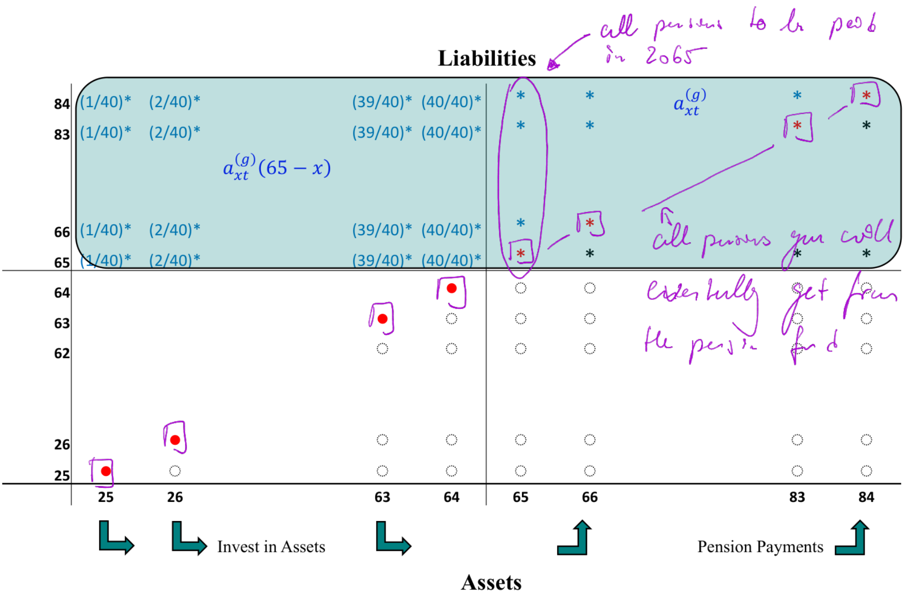
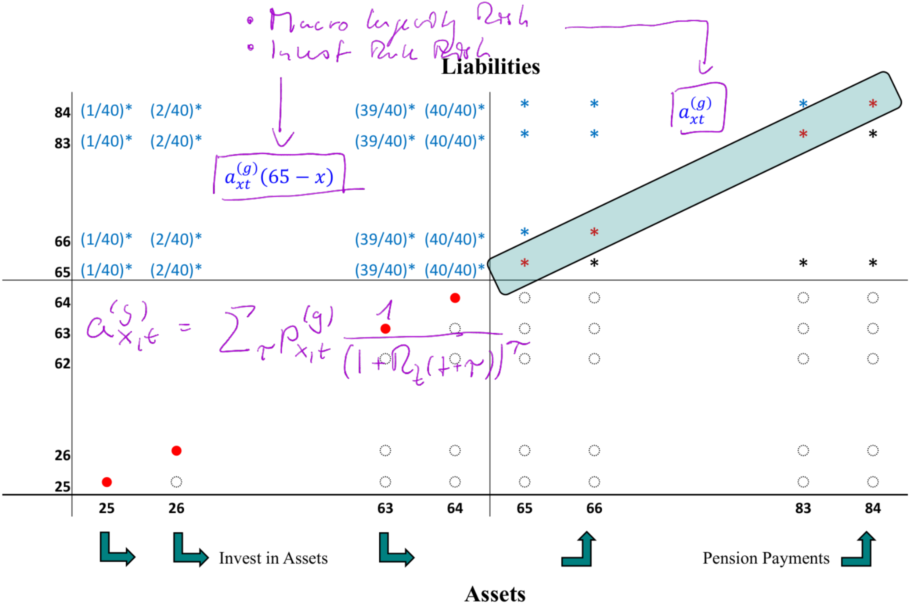


Contributions, Entitlements, and Pensions



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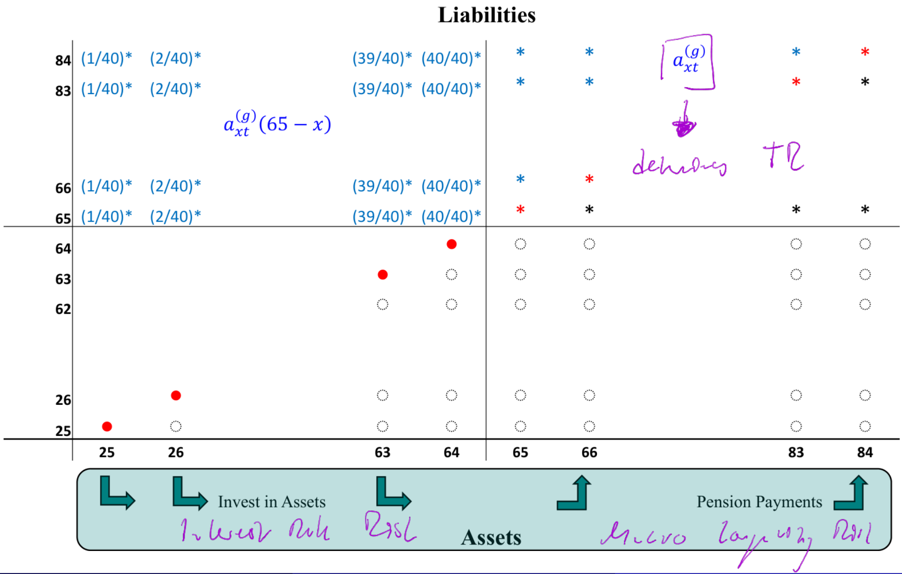

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- Balance sheet of a pension fund:

Assets	Liabilities
Cash	Equity
Bonds	
Stocks	Debt
Derivatives	◦ mortgages, credits, bonds
Real Estate	◦ technical reserves
	↳ liabilities to be made to its pensioneers

- Funding Ratio (FR) = Assets/Liabilities

↙ technical reserves

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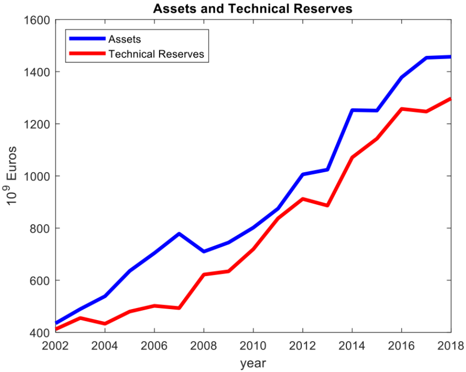
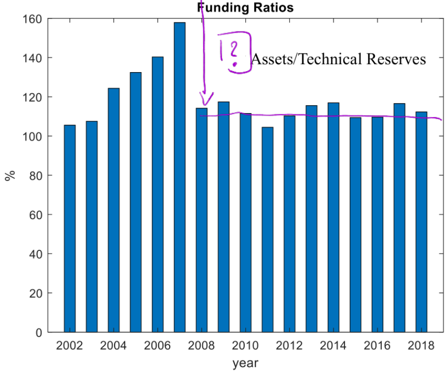
$FR = \frac{\text{Assets}}{\text{Techn. Reserves}}$

Financial Crisis



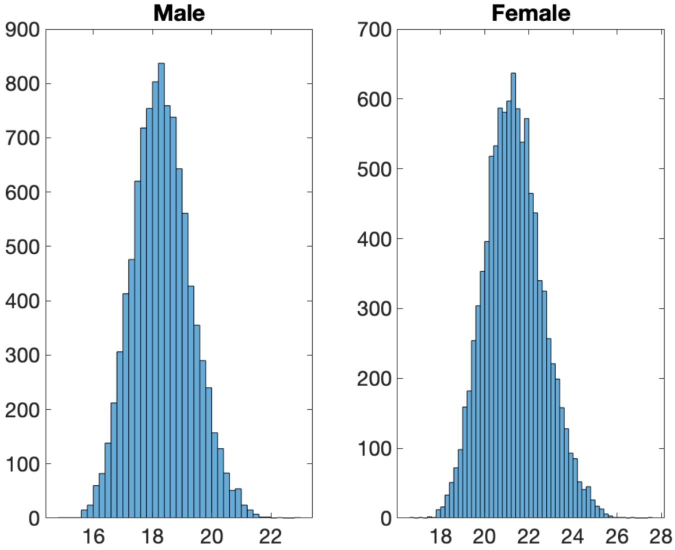


2008: stock market ↓ ⇒ A ↓ ⇒ FR ↓

↳ expansionary monetary policy

↳ interest rates ↓ ⇒ A ↑, $a_{x+t}^{(s)}$ ↑ ⇒ TR ↑

Simulation: Annuity Factor (see Part IV)



Simulation: Funding Ratios (see Part IV)

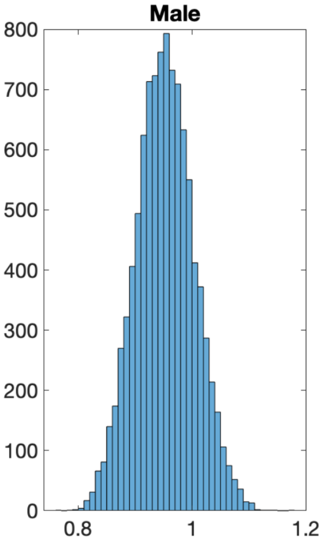


Table of Contents

1 Introduction

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6 Model Risk: A Very Brief Introduction

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- Let $T_{x,t}^{(g)}$ denote the (random) remaining lifetime of somebody of age x, group g, at time t with probability distribution function (cdf)

  age

$$F_{x,t}^{(g)}(\tau) = \mathbb{P}(T_{x,t}^{(g)} \leq \tau).$$

- Assuming that the remaining lifetime has a probability density (pdf) function $f_{x,t}^{(g)}$ such that $\frac{d}{d\tau} F_{x,t}^{(g)}(\tau) = f_{x,t}^{(g)}(\tau)$

$$\Longrightarrow \mathbb{P}(\tau < T_{x,t}^{(g)} < \tau + \mathrm{d}\tau) = f_{x,t}^{(g)}(\tau)\mathrm{d}\tau.$$

- τ-years survival probability of an individual of age x:

$$_{\tau}p_{x,t}^{(g)} = 1 - F_{x,t}^{(g)}(\tau)$$

$$p_{x,t}^{(g)} = {}_{1}p_{x,t}^{(g)}$$

Remaining Lifetime

$$\mathbb{P}\left(\tau \leq T_{x,t}^{(g)} \leq \tau + \Delta\tau \right)$$

$$= F_{x,t}^{(g)}(\tau + \Delta\tau) - F_{x,t}^{(g)}(\tau) \qquad \Big| : \Delta\tau \quad \Big| \lim_{\Delta\tau \to 0}$$

$$\lim_{\Delta\tau \to 0} \frac{F_{x,t}^{(g)}(\tau + \Delta\tau) - F_{x,t}^{(g)}(\tau)}{\Delta\tau} = \frac{d}{d\tau} F_{x,t}^{(g)}(\tau)$$

$$= f_{x,t}^{(g)}(\tau)$$

$$\Rightarrow \boxed{d F_{x,t}^{(g)}(\tau) = f_{x,t}^{(g)}(\tau)\, d\tau}$$

- The force of mortality (or hazard rate of death) is defined as

$$h^{(g)}_{x,t}(\tau) = \mu^{(g)}_{x+\tau,t+\tau} = -\frac{\partial}{\partial \tau} \ln\left({}_\tau p^{(g)}_{x,t}\right)$$

p. 55 →

representing the instantaneous rate of mortality at a certain age measured on an annualized basis.

- Formally,

$f_{x,t} = \mu_{x,t} \cdot (1 - F_{x,t})$

$$\mu^{(g)}_{x+\tau,t+\tau} = \frac{f^{(g)}_{x,t}(\tau)}{1 - F^{(g)}_{x,t}(\tau)}.$$

$\Rightarrow f^{(g)}_{x,t} = \mu^{(g)}_{x+\tau,t+\tau} \cdot (1 - F^{(g)}_{x,t}(\tau))$

- The instantaneous survival probability can be rewritten in terms of the hazard rate

$$\mathbb{P}(\tau < T^{(g)}_{x,t} < \tau + \mathrm{d}\tau) = f^{(g)}_{x,t}(\tau)\mathrm{d}\tau = \mu^{(g)}_{x+\tau,t+\tau} \cdot {}_\tau p^{(g)}_{x,t}\mathrm{d}\tau$$

Force of Mortality: Intuition

$${}_\tau p_{x,t}^{(g)} \in (0,1) \quad \forall x, t, g$$

$$\Rightarrow \exists\, {}_\tau M_{x,t}^{(g)} : \quad {}_\tau p_{x,t}^{(g)} = e^{-{}_\tau M_{x,t}^{(g)}}$$

$$\Downarrow \ln$$

$${}_\tau M_{x,t}^{(g)} = \int_0^\tau h_{x,t}^{(g)}(s)\, ds \qquad \ln {}_\tau p_{x,t}^{(g)} = -{}_\tau M_{x,t}^{(g)}$$

$$\Downarrow \frac{\partial}{\partial \tau}$$

$$\boxed{-\frac{\partial}{\partial \tau} \ln {}_\tau p_{x,t}^{(g)} = h_{x,t}^{(g)}(\tau)}$$

⚠ We use the hazard rate of death / force of mortality to write survival probabilities as exponential functions

Force of Mortality

- Integrating the force of mortality $\mu^{(g)}_{x+\tau,t+\tau} = -\frac{\partial}{\partial\tau}\ln({}_{\tau}p^{(g)}_{x,t})$ yields

$$\int_0^s \mu^{(g)}_{x+\tau,t+\tau}\mathrm{d}\tau = -\int_0^s \frac{\partial}{\partial\tau}\ln({}_{\tau}p^{(g)}_{x,t})\mathrm{d}\tau = -\ln({}_{s}p^{(g)}_{x,t}).$$

${}_sM^{(g)}_{x,t} =$

- Consequently,

$${}_{s}p^{(g)}_{x,t} = \exp\Big(-\int_0^s \mu^{(g)}_{x+\tau,t+\tau}\mathrm{d}\tau\Big).$$

- Assumption: $\mu^{(g)}_{x+\tau,t+\tau} = \mu^{(g)}_{x,t}$ if $0 \le \tau < 1$. (A)
- Under this assumption, the survival probability is

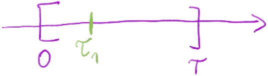
$${}_{s}p^{(g)}_{x,t} = \exp\big(-s\cdot\mu^{(g)}_{x,t}\big).$$

$s \in [0,1]$

- In particular, the one-year survival probability can be rewritten as

$$p^{(g)}_{x,t} = {}_{1}p^{(g)}_{x,t} = \exp\big(-\mu^{(g)}_{x,t}\big).$$

- Let $\tau = \tau_1 + \tau_2$. Then, the τ-years survival probability is given by



$$ {}_{\tau}p^{(g)}_{x,t} = {}_{\tau_1}p^{(g)}_{x,t} \cdot {}_{\tau_2}p^{(g)}_{x+\tau_1,t+\tau_1}. $$

- Proof:

$$ {}_{\tau}p^{(g)}_{x,t} = e^{-\int_0^{\tau} \mu^{(g)}_{x+s,t+s}\,ds} = e^{-\int_0^{\tau_1+\tau_2} \mu^{(g)}_{x+s,t+s}\,ds} $$

$$ = e^{-\int_0^{\tau_1} \mu^{(g)}_{x+s,t+s}\,ds - \int_{\tau_1}^{\tau_1+\tau_2} \mu^{(g)}_{x+s,t+s}\,ds} $$

$$ = e^{-\int_0^{\tau_1} \mu^{(g)}_{x+s,t+s}\,ds} \cdot e^{-\int_{\tau_1}^{\tau_1+\tau_2} \mu^{(g)}_{x+s,t+s}\,ds} $$

$$ = {}_{\tau_1}p^{(g)}_{x,t} \cdot {}_{\tau_2}p^{(g)}_{x+\tau_1,t+\tau_1} \qquad \square $$

$$ e^{x+y} = e^x \cdot e^y $$

Decomposition

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- In general, the τ-years survival probability can be decomposed into

$$ {}_{\tau}p_{x,t}^{(g)} = \prod_{k=0}^{\tau-1} p_{x+k,t+k}^{(g)}, \qquad p_{x,t}^{(g)} = 1 - q_{x,t}^{(g)}. $$

- Here, the one-year survival and death probabilities can be rewritten as

$$ \hat{p}_{x,t}^{(g)} = \exp(-\hat{\mu}_{x,t}^{(g)}), $$
$$ \hat{q}_{x,t}^{(g)} = 1 - \exp(-\hat{\mu}_{x,t}^{(g)}). $$

- **Moral**: Modeling the force of mortality is sufficient to model the survival and death probabilities.

  → How to estimate the historical force of mortality? → Maximum Likelihood (ML)

  → How to model the evolution of the force of mortality?

  → Lee-Carter model / AG-model

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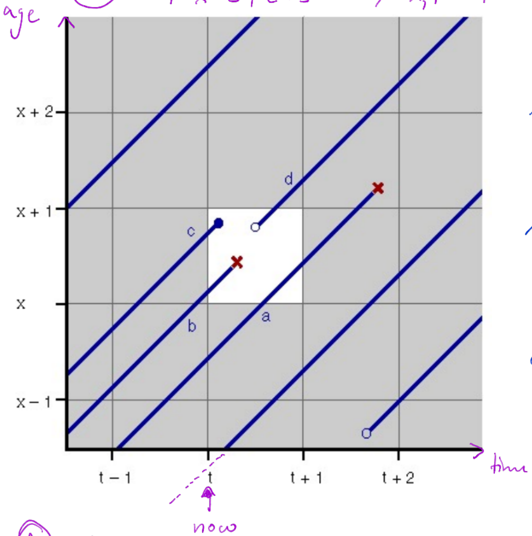
(A) $\mu^{(g)}_{x+s,t+s} = \mu_{x,t}$, $s \le 1$



(filled dot) individual has left the population

(cross) individual has passed away

(open dot) baby was born or an individual has entered the population

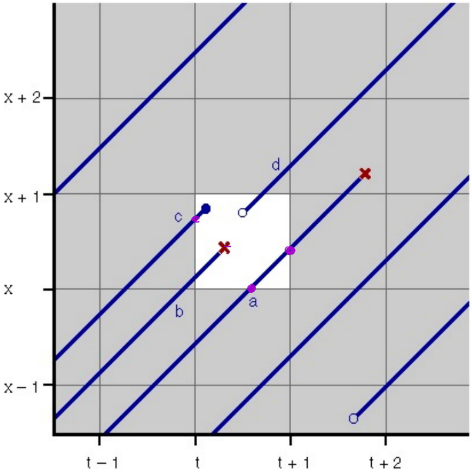
(A) hazard rate of death within one square is constant.

- Let $n_{x,t}^{(g)}$ lives contribute to the observations in the white square (Lexis diagram).
- Assume life $i \in N_{x,t}^{(g)} = \{1, \ldots, n_{x,t}^{(g)}\}$ is observed between the ages of $x + t_i$ and $x + s_i$ (with $0 \leq t_i < s_i \leq 1$).
- The *exposure* $E_{x,t}^{(g)}$ is defined as

$$E_{x,t}^{(g)} = \sum_{i=1}^{n_{x,t}^{(g)}} (s_i - t_i).$$

- $D_{x,t}^{(g)} \in \mathbb{N}$ denotes the *number of deaths* observed in the white square.
- $I_{x,t}^{(g)} \subseteq N_{x,t}^{(g)}$ is the subset of observations i terminated by death.

Example: Lexis Diagram



$n^{(g)}_{x,t} = 4$, $N^{(g)}_{x,t} = \{1,2,3,4\}$

a) $t_1 = 0$, $s_1 = 0.4$

b) $t_1 = 0.2$, $s_2 = 0.5$

c) $t_3 = 0.8$, $s_3 = 0.9$

d) $t_4 = 0.9$, $s_4 = 1$

$$E^{(g)}_{x,t} = \sum_{i=1}^{n^{(g)}_{x,t}} (s_i - t_i)$$

$$= 0.4 + 0.3 + 0.1 + 0.1$$

$$= \underline{\underline{0.9}}$$

$D^{(g)}_{x,t} = 1$, $I^{(g)}_{x,t} = \{2\}$

$f^{(g)}_{x,t}$: pdf of $T^{(g)}_{x,t}$; $F^{(g)}_{x,t}$: cdf of $T^{(g)}_{x,t}$

- Assume that the $n^{(g)}_{x,t}$ lives are independent, i.e., their remaining lifetimes $T^{(g)}_{x,t}$ are stochastically independent.
- The likelihood of the $n^{(g)}_{x,t}$ observations is

first $D^{(g)}_{x,t}$ individuals; $N^{(g)}_{x,t} - D^{(g)}_{x,t}$ individuals

$$L^{(g)}_{x,t} = \prod_{i \in I^{(g)}_{x,t}} f^{(g)}_{x+t_i,t}(s_i - t_i) \prod_{i \notin I^{(g)}_{x,t}} \left(1 - F^{(g)}_{x+t_i,t}(s_i - t_i)\right)$$

passed away; survivors

$$= \prod_{i \in I^{(g)}_{x,t}} \mu^{(g)}_{x+s_i,t+s_i-t_i} \cdot {}_{s_i-t_i}p^{(g)}_{x+t_i,t} \prod_{i \notin I^{(g)}_{x,t}} {}_{s_i-t_i}p^{(g)}_{x+t_i,t} \cdot$$

$\mu^{(g)}_{x,t}$

- Using our standing assumption $\mu^{(g)}_{x+s_i,t+s_i} = \mu^{(g)}_{x,t},\ s_i, t_i \in [0,1]$:

$${}_{s_i-t_i}p^{(g)}_{x+t_i,t} = \exp\left(-(s_i - t_i)\mu^{(g)}_{x+s_i,t+s_i-t_i}\right) = \exp\left(-(s_i - t_i)\mu^{(g)}_{x,t}\right)$$

- Consequently, the likelihood can be expressed in terms of $\mu_{x,t}^{(g)}$:

$$L_{x,t}^{(g)} = \prod_{i \in I_{x,t}^{(g)}} \mu_{x,t}^{(g)} \exp\left(-(s_i - t_i)\mu_{x,t}^{(g)} \right) \prod_{i \notin I_{x,t}^{(g)}} \exp\left(-(s_i - t_i)\mu_{x,t}^{(g)} \right)$$

$$= \prod_{i \in I_{x,t}^{(g)}} \mu_{x,t}^{(g)} \prod_{i \in N_{x,t}^{(g)}} \exp\left(-(s_i - t_i)\mu_{x,t}^{(g)} \right)$$

← all individuals

$$= \left[\mu_{x,t}^{(g)}\right]^{D_{x,t}^{(g)}} \exp\left(-\sum_{i \in N_{x,t}^{(g)}} (s_i - t_i)\mu_{x,t}^{(g)} \right).$$

$$= e^{-\mu_{x,t} \sum_{i \in N_{x,t}^{(g)}} (s_i - t_i)}$$

- Recall: exposure is defined as $E_{x,t}^{(g)} = \sum_{i=1}^{n_{x,t}^{(g)}} (s_i - t_i)$, i.e.,

$$L_{x,t}^{(g)} = \left[\mu_{x,t}^{(g)}\right]^{D_{x,t}^{(g)}} \exp\left(-E_{x,t}^{(g)} \mu_{x,t}^{(g)} \right).$$

Maximum Likelihood Estimate

- Maximizing the likelihood function

$$L_{x,t}^{(g)} = \left[\mu_{x,t}^{(g)}\right]^{D_{x,t}^{(g)}} \exp\left(- E_{x,t}^{(g)} \mu_{x,t}^{(g)} \right)$$

w.r.t. $\mu_{x,t}^{(g)}$ results in the maximum likelihood estimate

ML-estimator:

$$\widehat{\mu}_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}. \quad = m_{x,t}^{(g)}$$

- The ratio $\frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}$ is called the *raw central death rate* and usually denoted by $m_{x,t}^{(g)}$.

Proof: Maximum Likelihood Estimate

$$L_{x,t}^{(g)} = \left(\mu_{x,t}^{(g)}\right)^{D_{x,t}^{(g)}} e^{-\mu_{x,t}^{(g)} \cdot E_{x,t}^{(g)}} \quad \Big| \log$$

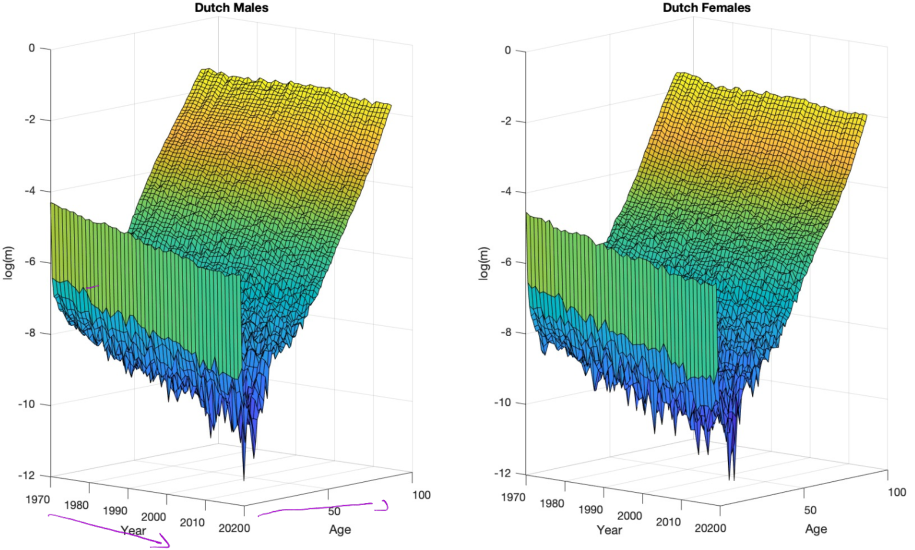
$$\ell_{x,t}^{(g)} = D_{x,t}^{(g)} \ln\left(\mu_{x,t}^{(g)}\right) - \mu_{x,t}^{(g)} E_{x,t}^{(g)} \quad \Big| \frac{\partial}{\partial \mu_{x,t}^{(g)}}$$

$$\frac{D_{x,t}^{(g)}}{\mu_{x,t}^{(g)}} - E_{x,t}^{(g)} = 0$$

$$\Rightarrow \boxed{\hat{\mu}_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}}$$

ML-estimator

Central Death Rates



1. For past and present periods, we estimate

$$\widehat{p}_{x,t}^{(g)} = \exp(-m_{x,t}^{(g)}),$$
$$\widehat{q}_{x,t}^{(g)} = 1 - \exp(-m_{x,t}^{(g)})$$

with $m_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}$.

2. For future periods, $D_{x,t+\dots}$ and $E_{x,t+\dots}$ are unobserved, and we need *models* to predict the future probabilities

$$p_{x,t}^{(g)} = \exp(-\mu_{x,t}^{(g)}),$$
$$q_{x,t}^{(g)} = 1 - \exp(-\mu_{x,t}^{(g)}).$$

- Lee & Carter (1992) and others: directly model $m_{x,t}^{(g)}$.
- AG2022 and others: model $D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}(\mu_{x,t}^{(g)} E_{x,t}^{(g)})$.

Lee-Carter Model

$$
\begin{array}{c}
\rightarrow \text{time} \\
\downarrow \text{age}
\end{array}
\begin{pmatrix}
\hat{q}^{(g)}_{0,t_0} & \hat{q}^{(g)}_{0,t_0+1} & \cdots & \hat{q}^{(g)}_{0,t_0+T} & \cdots \\
\hat{q}^{(g)}_{1,t_0} & \hat{q}^{(g)}_{1,t_0+1} & \cdots & \hat{q}^{(g)}_{1,t_0+T} & \cdots \\
\hat{q}^{(g)}_{2,t_0} & \hat{q}^{(g)}_{2,t_0+1} & \cdots & \hat{q}^{(g)}_{2,t_0+T} & \cdots \\
\vdots & & & \vdots & \\
\vdots & & & \vdots & \\
\hat{q}^{(g)}_{x_{\max},t_0} & \hat{q}^{(g)}_{x_{\max},t_0+1} & \cdots & \hat{q}^{(g)}_{x_{\max},t_0+T} & \cdots
\end{pmatrix}
$$

→ *Predictions*

→ *Predictions*

→ *Predictions*

→ *Predictions*

← *Observed* →

$$\hat{q}^{(g)}_{xt} = 1 - \exp\left(-m^{(g)}_{x,t}\right)$$

Model

Table of Contents

1 Introduction

2 Relevance of Macro Longevity Risk
- First Pillar: AOW
- Second Pillar: Pension Funds

3 Modeling Mortality

4 Benchmark Model
- The Lee-Carter Model
- Alternative Estimation
- Some Applications and Extensions

5 The AG2022 Model and COVID-19
- Model and Projections
- Closure of the Life Table

6 Model Risk: A Very Brief Introduction

The Lee & Carter (1992) Model

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- Lee and Carter (1992) directly model the evolution of the central death rate $m_{x,t}^{(g)}$ in a parsimonious way.
- Death rate evolves stochastically according to the dynamics

age-fixed effects — dynamic age component

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)},$$

noise

age-specific sensitivity — the trend of central death rates

where $\varepsilon_{x,t}^{(g)} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\varepsilon^{(g)}})$. $\alpha_x^{(g)}$ and $\beta_x^{(g)}$ are age-specific parameters.
- Dynamics of the latent factor, typically modeled as a random walk with drift:

$$\Delta \kappa_t^{(g)} = c^{(g)} + \delta_t^{(g)}$$

$$\kappa_t^{(g)} = c^{(g)} + \kappa_{t-1}^{(g)} + \delta_t^{(g)},$$

where $\delta_t^{(g)} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\delta^{(g)}})$.

Given the estimates of $\widehat{\alpha}_x^{(g)}$, $\widehat{\beta}_x^{(g)}$, and $\widehat{\kappa}_t^{(g)}$, we can forecast the best estimates of the death rates, and hence the death probabilities:

$$\ln(\widehat{m}_{x,T+t_i}^{(g)}) = \widehat{\alpha}_x^{(g)} + \widehat{\beta}_x^{(g)} \widehat{\kappa}_{T+t_i}^{(g)}, \qquad i = 1, \dots, N$$

Consequently,

$$\widehat{p}_{x,T+t_i}^{(g)} = \mathrm{e}^{-\widehat{m}_{x,T+t_i}^{(g)}}.$$

1. Original approach: Lee & Carter (1992):
   - Estimation of $\alpha_x^{(g)}$, $\beta_x^{(g)}$, $\kappa_t^{(g)}$ by *singular value decomposition* (SVD).
2. Iterative minimization of sum of squared errors:

$$\min_{\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}} \sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} - \beta_x^{(g)} \kappa_t^{(g)} \right]^2$$

- The input to the model is a matrix of age-specific mortality rates (a life table of group g).
  - Let $\mathcal{X}$ be a set of size X of included ages, e.g., $\mathcal{X} = \{0, \ldots, 90\}$, $X = 91$.
  - Let $\mathcal{T}$ be the set of size T of included periods, e.g., $\mathcal{T} = \{1970, \ldots, 2021\}$, $T = 52$.
- **First step:** estimate $\alpha_x^{(g)}$ as the average over time of $\ln(m_{x,t}^{(g)})$, $x \in \mathcal{X}$, $t \in \mathcal{T}$:

$$\widehat{\alpha}_x^{(g)} = \frac{1}{T} \sum_{t=1}^{T} \ln(m_{x,t}^{(g)})$$

- **Second step:** Calculate the matrix

$$M^{(g)} = \left(\ln(m_{x,t}^{(g)}) - \widehat{\alpha}_x^{(g)}\right)_{x \in \mathcal{X}, t \in \mathcal{T}} \in \mathbb{R}^{X \times T}$$

Estimation: First step

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$$\min_{\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}} \sum_{\substack{x \in \mathcal{X} \\ t \in \mathcal{T}}} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} - \beta_x^{(g)} \kappa_t^{(g)} \right]^2$$

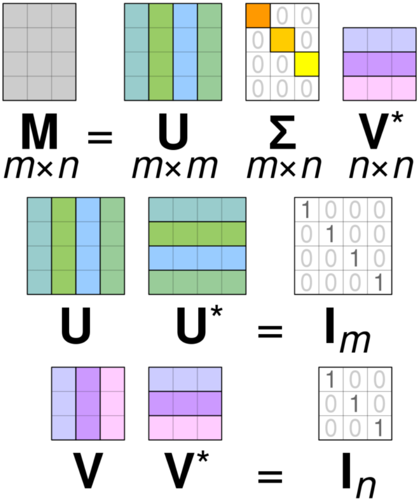
$$0 = \frac{\partial}{\partial \alpha_x} \sum_{x,t} [\dots]^2 = -2 \sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} - \beta_x^{(g)} \kappa_t^{(g)} \right]$$

Lee-Carter: $\sum_x \beta_x^{(g)} = 1, \quad \frac{1}{T} \sum_t \kappa_t^{(g)} = 0$

$$\Rightarrow -2 \sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} \right] = 0$$

$$\Rightarrow \hat{\alpha}_x^{(g)} = \frac{1}{T} \sum_t \ln(m_{x,t}^{(g)})$$

Normalization is important for identification.



https://commons.wikimedia.org/w/index.php?curid=67853297

$M^{(g)}$ real matrix

$M^{(g)} = U \cdot \Sigma \cdot V$

$UU' = I_X = U'U$

$VV' = I_T = V'V$

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \vdots & 0 \\ & \ddots & & \vdots & \vdots \\ & & \sigma_T & \vdots & 0 \\ \hline 0 & \cdots & 0 & \vdots & 0 \end{pmatrix}$$

$$\min_{A} \| M - A \|_F \quad \text{s.t.: } \text{rk}(A) \leq 1$$

$$M = U \Sigma V \quad (\text{SVD})$$

Σ_{11} denotes the largest singular value

$$\hat{A} = \underbrace{u_1}_{\text{first column of } U} \cdot \Sigma_{11} \cdot \underbrace{v_1'}_{\text{first column of } V}$$

Estimation: Original Approach – SVD

- **Third step:** Singular value decomposition can be performed numerically.
- Applying the singular value decomposition to the matrix

$$M^{(g)} = \left(\ln(m_{x,t}^{(g)}) - \widehat{\alpha}_x^{(g)} \right)_{x \in \mathcal{X}, t \in \mathcal{T}} \in \mathbb{R}^{X \times T}$$

  yields three matrices $U^{(g)} \in \mathbb{R}^{X \times X}$, $\Sigma^{(g)} \in \mathbb{R}^{X \times T}$, and $V^{(g)} \in \mathbb{R}^{T \times T}$ such that

$$M^{(g)} = U^{(g)} \Sigma^{(g)} V^{(g)\prime}.$$

- Unlike standard OLS estimation, this method is not plagued by the existence of multiple local minima. According to the Eckart-Young-Mirsky Theorem, the global minima can be directly obtained from the SVD decomposition.

Estimation: Original Approach – SVD

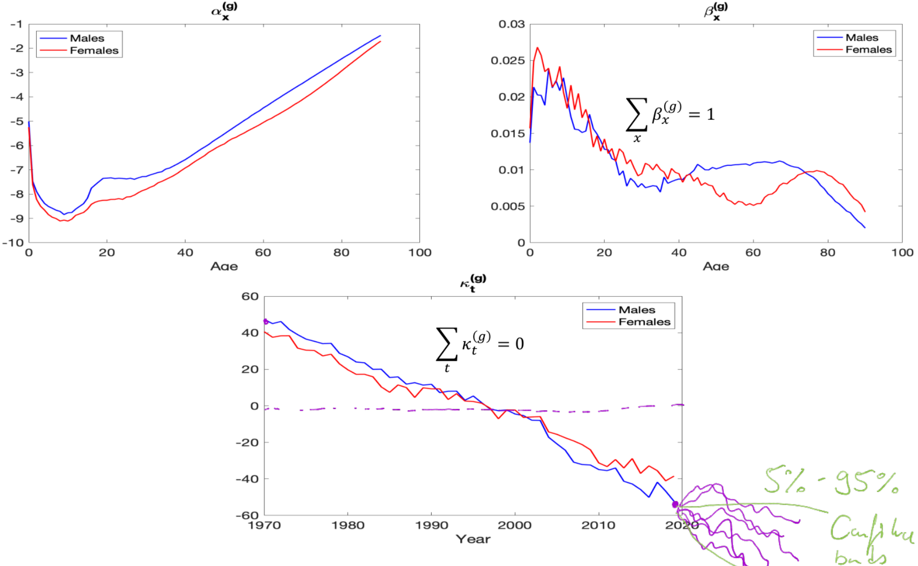
- **Fourth step:** Calculate the estimates $\widehat{\beta}_x^{(g)}$, $\widehat{\kappa}_t^{(g)}$ from $U^{(g)} = (u_1^{(g)}, \ldots, u_X^{(g)}) \in \mathbb{R}^{X \times X}$ and $V^{(g)} = (v_1^{(g)}, \ldots, v_T^{(g)}) \in \mathbb{R}^{T \times T}$.
- Skipping all the technical details, one obtains under appropriate normalizations ($\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t \in \mathcal{T}} \kappa_t^{(g)} = 0$) the estimates

$$\widehat{\beta}_x^{(g)} = u_1^{(g)},$$
$$\widehat{\kappa}_t^{(g)} = \Sigma_{1,1}^{(g)} v_1^{(g)},$$

  where $\Sigma_{1,1}^{(g)}$ is the largest singular value.
- *Remark*: The original SVD approach is equivalent to LS estimation. We will discuss this in the tutorials.

↳ Problem #9.

Estimates


$\alpha_x^{(g)}$
Males
Females
Age
$\beta_x^{(g)}$
Males
Females
$\sum_x \beta_x^{(g)} = 1$
Age
$\kappa_t^{(g)}$
Males
Females
$\sum_t \kappa_t^{(g)} = 0$
Year
5% - 95%

- **Fifth step:** Simulate $\kappa_{T+t_i}^{(g)}$, $i = 1, \ldots, N$, i.e., for N additional years.
- One has to specify the dynamics of the time trend $\kappa_t^{(g)}$.
- Well established and typically a very good fit:

$$\kappa_t^{(g)} = c^{(g)} + \kappa_{t-1}^{(g)} + \delta_t^{(g)}$$

Consequently, the time trend evolves like a random walk with drift

$$\Delta\kappa_t^{(g)} = c^{(g)} + \delta_t^{(g)}.$$

$\kappa_t^{(g)} - \kappa_{t-1}^{(g)}$

- Estimation:

$$\widehat{c}^{(g)} = \frac{1}{T-1}\sum_{t=2}^{T}\Delta\kappa_t^{(g)} = \frac{\kappa_T^{(g)} - \kappa_1^{(g)}}{T-1}.$$

Forecasting

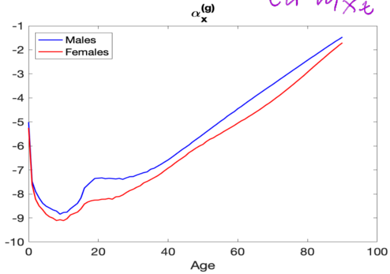
- **Sixth step:** Use the simulated data to predict the best estimates of the future death and survival probabilities and to estimate confidence intervals for these variables.
- Perform a Monte-Carlo simulation for $\kappa_{T+t_i}^{(g)}$ and $m_{x,T+t_i}^{(g)}$, i.e., simulate a large number of paths $\omega \in \Omega$, say $\mid \Omega \mid = 10,000$.
- Compute for each path $\omega \in \Omega$ the survival and death probabilities

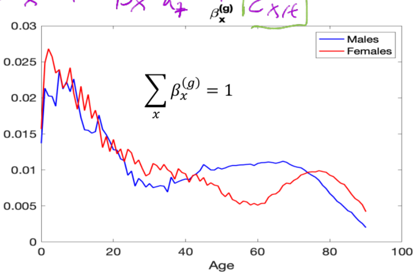
$$\widehat{p}_{x,T+t_i}^{(g)}(\omega) = \mathrm{e}^{-\widehat{m}_{x,T+t_i}^{(g)}(\omega)}, \qquad \widehat{q}_{x,T+t_i}^{(g)}(\omega) = 1 - \widehat{p}_{x,T+t_i}^{(g)}(\omega).$$

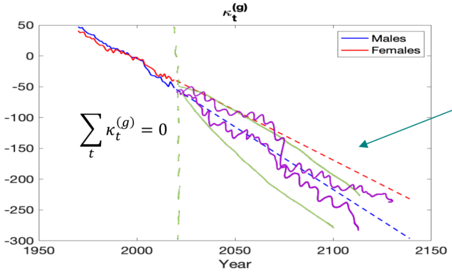
- Make a probability distribution (e.g., a histogram) for the forecasted probabilities.
- Derive the relevant moments from the resulting distribution such as mean, median, standard deviation, skewness, 5% and 95%-quantile,. . . .

Best Estimate Forecast

$\ln m_{x,t}^{(g)} = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}$

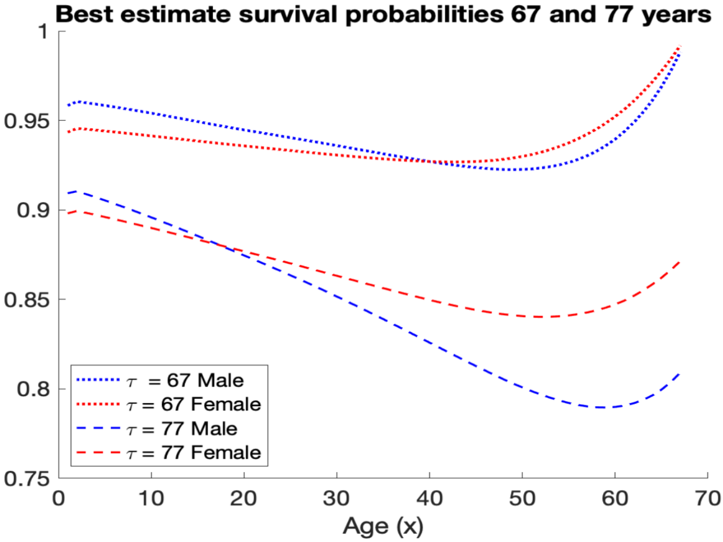




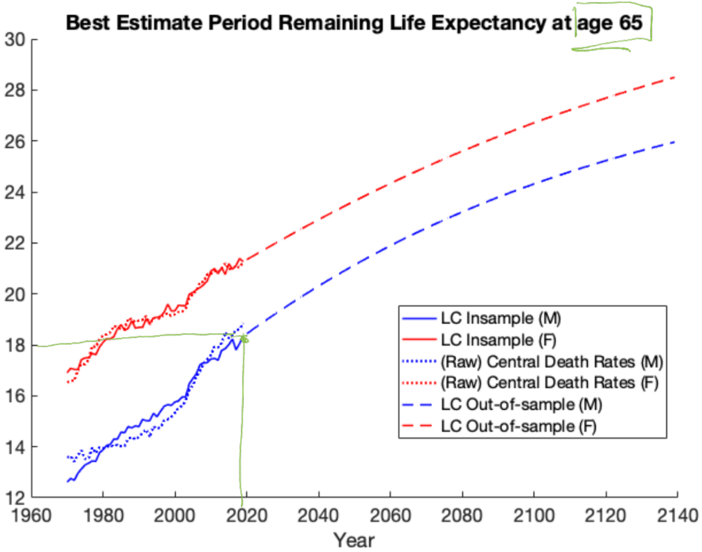


Best Estimates:

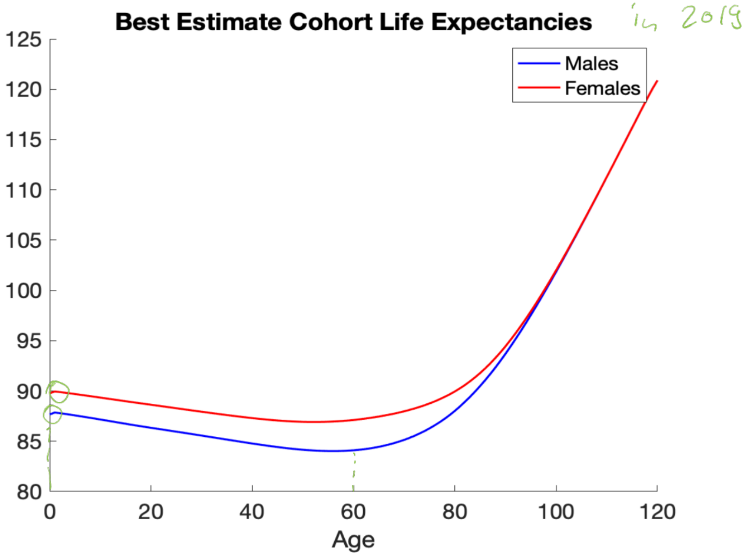
Best Estimate Survival Probabilities



Best Estimate Period Life Expectancy



Best Estimate Cohort Life Expectancy



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- We now consider future periods:

$$\Delta\kappa_t^{(g)} = c^{(g)} + \delta_t^{(g)}$$

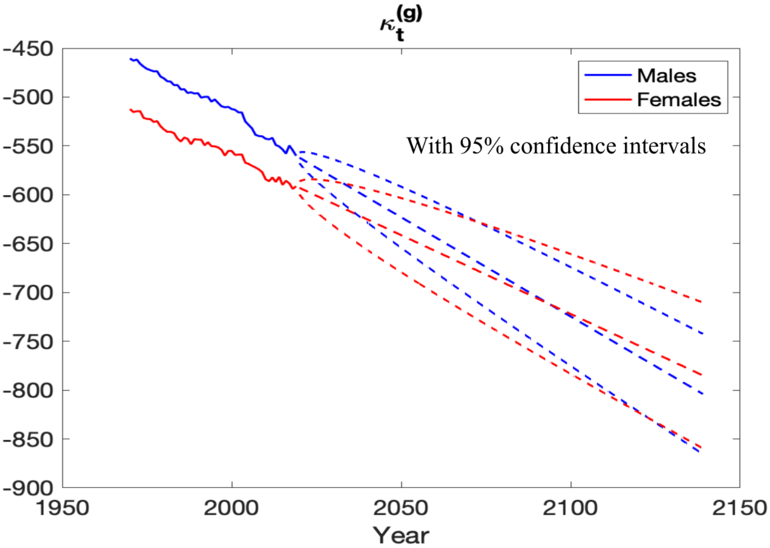
- These dynamics imply

$$\kappa_{T+t_i}^{(g)} = \underbrace{\kappa_T^{(g)} + t_i \cdot c^{(g)}}_{\text{Best Estimate}} + \underbrace{\sum_{j=1}^{i} \delta_{T+t_j}^{(g)}}_{\text{Forecast Error}} .$$

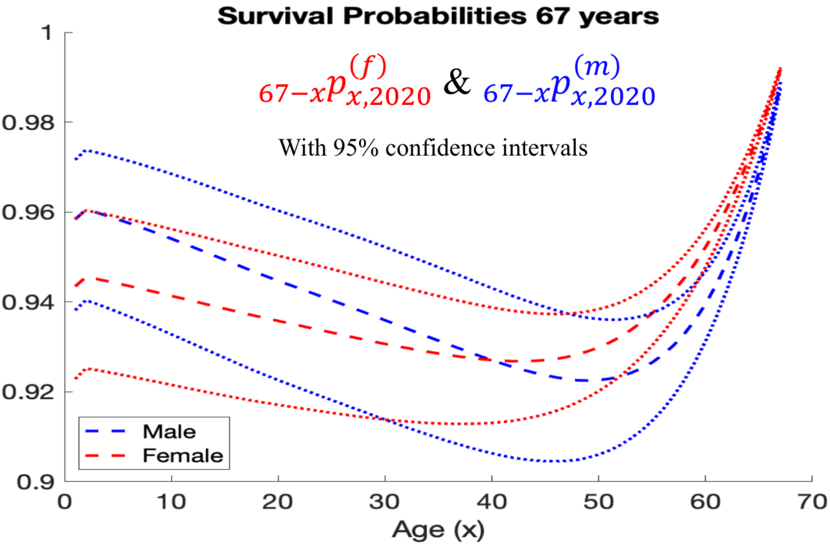
- Because $\delta_t^{(g)} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\delta(g)})$, the distribution of the trend component is

$$\kappa_{T+t_i}^{(g)} \sim \mathcal{N}\left(\kappa_T^{(g)} + t_i \cdot c^{(g)}, t_i \cdot \sigma^2_{\delta(g)}\right).$$

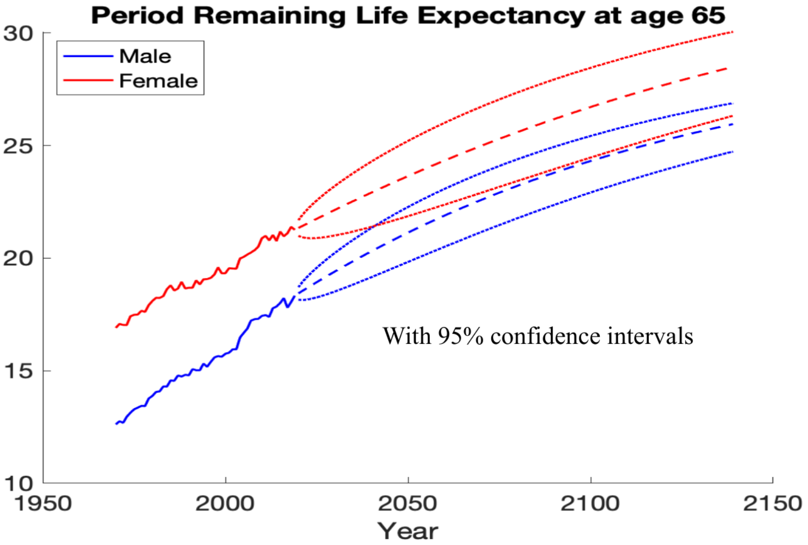
Macro Longevity Risk – Forecast



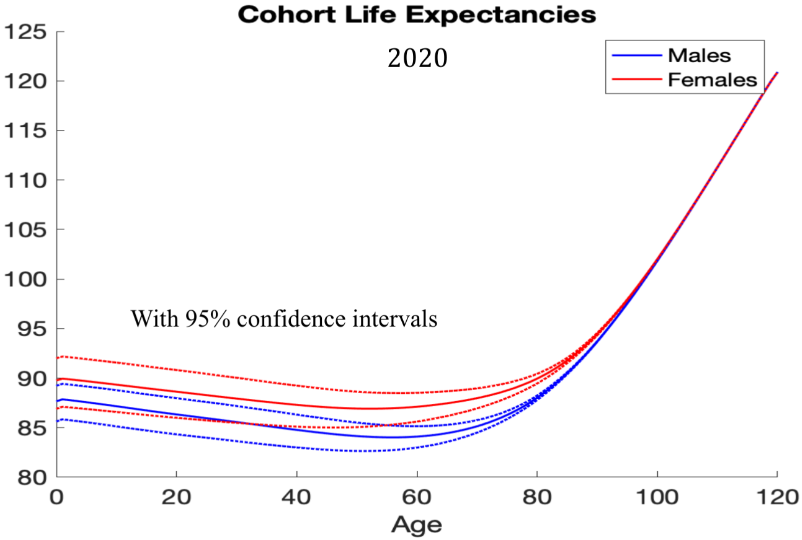
Macro Longevity Risk – Survival Probabilities



Macro Longevity Risk – Period Life Expectancy



Macro Longevity Risk – Cohort Life Expectancy



Ordinary Least Squares

- Assume the Lee-Carter normalization ($\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t\in\mathcal{T}} \kappa_t^{(g)} = 0$).
- In Problem 9, it is to show that the estimation of $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$ by the Singular Value Decomposition (SVD) is the same as minimizing the sum of squared errors

$$\min_{\alpha_x^{(g)},\beta_x^{(g)},\kappa_t^{(g)}} \sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} - \beta_x^{(g)}\kappa_t^{(g)}\right]^2$$

w.r.t $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$.

- This minimization is typically done iteratively:
  - minimize w.r.t. $\alpha_x^{(g)}$ (all x), keeping all $\beta_x^{(g)}, \kappa_t^{(g)}$ fixed,
  - minimize w.r.t. $\beta_x^{(g)}$ (all x), keeping all $\alpha_x^{(g)}, \kappa_t^{(g)}$ fixed,
  - minimize w.r.t. $\kappa_t^{(g)}$ (all t), keeping all $\alpha_x^{(g)}, \beta_x^{(g)}$ fixed,
  - keep iterating until convergence.
- We will now dive deeper into this issue.

The Necessity of Normalizations

- If $\widehat{\alpha}_x^{(g)}, \widehat{\beta}_x^{(g)}, \widehat{\kappa}_t^{(g)}$ minimize the sum of squared errors

$$\sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \widehat{\alpha}_x^{(g)} - \widehat{\beta}_x^{(g)} \widehat{\kappa}_t^{(g)} \right]^2,$$

then also $\widetilde{\alpha}_x^{(g)} = \widehat{\alpha}_x^{(g)} + c_1 \widehat{\beta}_x^{(g)}$, $\widetilde{\beta}_x^{(g)} = c_2 \widehat{\beta}_x^{(g)}$, $\widetilde{\kappa}_t^{(g)} = \frac{\widehat{\kappa}_t^{(g)}}{c_2} - \frac{c_1}{c_2}$.

$$\tilde{\alpha}_x^{(g)} + \tilde{\beta}_x^{(g)} \tilde{\kappa}_t^{(g)} = \hat{\alpha}_x^{(g)} + c_1 \hat{\beta}_x^{(g)} + c_2 \hat{\beta}_x^{(g)} \left(\frac{\hat{\kappa}_t}{c_2} - \frac{c_1}{c_2} \right)$$

$$= \hat{\alpha}_x^{(g)} + \hat{\beta}_x^{(g)} \hat{\kappa}_t^{(g)}$$

- So, we need two normalizations. Standard normalizations: $\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t \in \mathcal{T}} \kappa_t^{(g)} = 0$.
- In the sequel we shall compare the standard normalization to the normalization of Liu et al. (2019a & b), i.e., $\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{x \in \mathcal{X}} \alpha_x^{(g)} = 0$ (see Problem 11).

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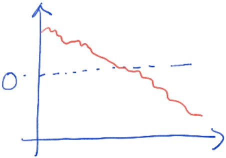
Lee-Carter

$$\sum_{t \in \mathcal{T}} \kappa_t^{(s)} = 0$$



- on average, the time effect is zero
- for the average age group, e.g. levels is captured by α_x

Li et al

$$\sum_{x \in \mathcal{X}} \alpha_x^{(g)} = 0$$

- avg age-fixed effect is zero
- $\kappa_t^{(g)}$ is the systematic factor also for the avg age group.

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- Choose starting values for $\beta_x^{(g)}, \kappa_t^{(g)}$ (with $\alpha_x^{(g)} = \frac{1}{T}\sum_{t=1}^{T} \ln(m_{x,t}^{(g)})$ and under the standard normalization $\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t\in\mathcal{T}} \kappa_t^{(g)} = 0$).
- Take average over x: fixed t

$$\frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}} \ln m_{x,t}^{(g)} = \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}}\left(\alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}\right)$$

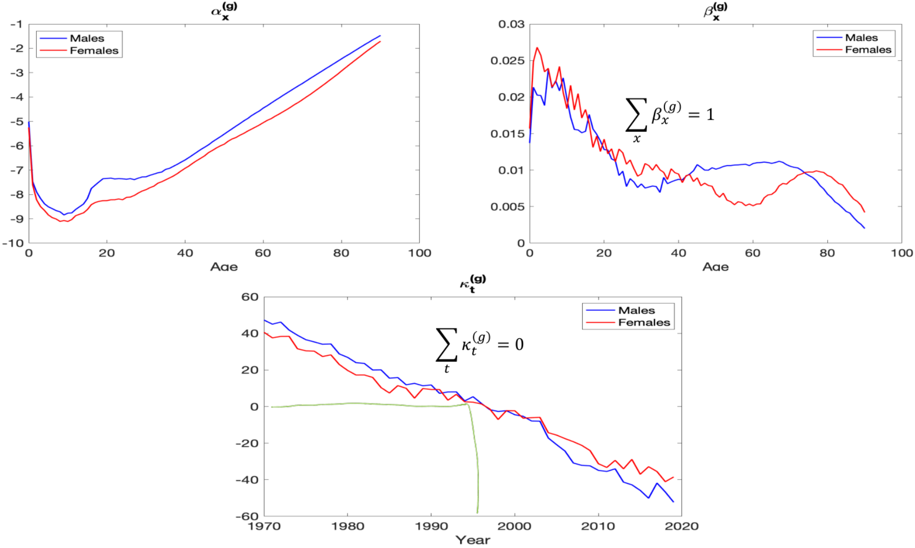
$$= \frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}}\alpha_x^{(g)} + \kappa_t^{(g)}\underbrace{\sum_{x\in\mathcal{X}}\beta_x^{(g)}}_{=1}\frac{1}{|\mathcal{X}|} + \underbrace{\frac{1}{|\mathcal{X}|}\sum_{x\in\mathcal{X}}\varepsilon_{x,t}^{(g)}}_{? \;\xrightarrow[LLN]{P}\; 0}$$

- Take time differences and then average over t:

$$\frac{1}{T-1}\sum_{t\in\mathcal{T}}\Delta\ln(m_{x,t}^{(g)}) = \frac{1}{T-1}\sum_t \Delta\left(\alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}\right) \xrightarrow{LLN} 0$$

$$= \underbrace{\frac{1}{T-1}\sum_t \Delta\alpha_x^{(g)}}_{=0} + \frac{1}{T-1}\sum_t \beta_x^{(g)}\Delta\kappa_t^{(g)} + \frac{1}{T-1}\sum_t \varepsilon_{x,t}$$

Estimates (Lee and Carter 1992)



- Liu et al. (2019a,b) propose an alternative normalization which makes estimation of the Lee-Carter model possible using linear regressions.
- Choose starting values for $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$ (under the Liu normalization $\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{x\in\mathcal{X}} \alpha_x^{(g)} = 0$).
- Take sum over x:

$$Z_t^{(g)} := \sum_{x\in\mathcal{X}} \ln(m_{x,t}^{(g)}) = \sum_{x\in\mathcal{X}} \underbrace{\alpha_x^{(g)}}_{=0} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}$$

$$= \kappa_t^{(g)} + \underbrace{\sum \varepsilon_{x,t}^{(g)}}_{\xrightarrow{LLN} 0}$$

- Run regressions for each x:

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}$$

$$\approx \alpha_x^{(g)} + \beta_x^{(g)} \cdot Z_t^{(g)} + \varepsilon_{xt}^{(g)}$$

- Start from the Lee-Carter model $\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}$ combined with the normalization $\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{x\in\mathcal{X}} \alpha_x^{(g)} = 0$.
- Define

$$\Rightarrow \quad Z_t^{(g)} = \sum_{x\in\mathcal{X}} \ln(m_{x,t}^{(g)}) = \sum_{x\in\mathcal{X}} \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}$$

$$= \kappa_t^{(g)} + \underbrace{\sum_{x\in\mathcal{X}} \varepsilon_{x,t}^{(g)}}_{= e_t^{(g)}} .$$

- Assume now a random walk with drift for $\kappa_t^{(g)}$:

$$\kappa_t^{(g)} = c^{(g)} + \kappa_{t-1}^{(g)} + \delta_t^{(g)}$$

$$\iff \quad Z_t^{(g)} = c^{(g)} + Z_{t-1}^{(g)} + \delta_t^{(g)} + e_t^{(g)} - e_{t-1}^{(g)}.$$

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- Next, we substitute

$$Z_t^{(g)} = c^{(g)} + Z_{t-1}^{(g)} + \left(\delta_t^{(g)} + e_t^{(g)} - e_{t-1}^{(g)}\right)$$

into

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}.$$

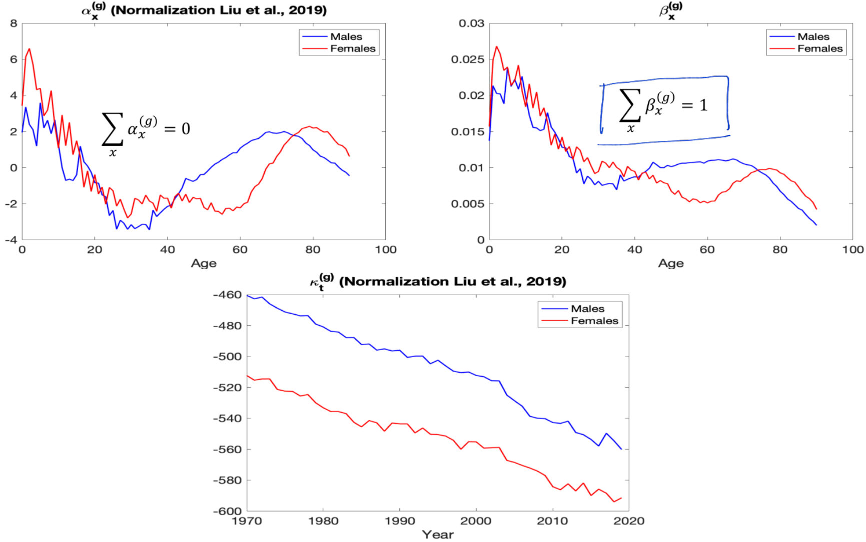
- Thus, we arrive at

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} Z_t^{(g)} + \left(\varepsilon_{x,t}^{(g)} - \beta_x^{(g)} e_t^{(g)}\right),$$

$$Z_t^{(g)} = c^{(g)} + Z_{t-1}^{(g)} + \left(\delta_t^{(g)} + e_t^{(g)} - e_{t-1}^{(g)}\right).$$

- The parameters in these equations can be estimated in a single step, applying standard linear regression techniques (possibly using instrumental variables).

Estimates (Liu et al. 2019)

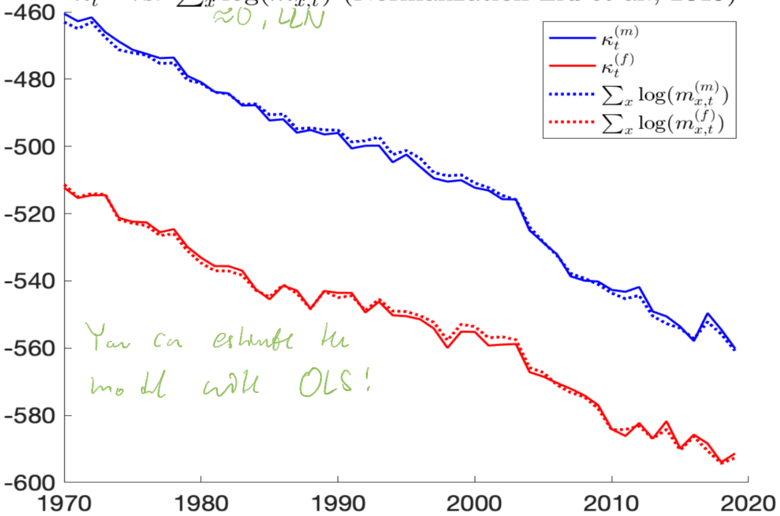


A Comparison

$\sum_x \ln m_{x,t}^{(g)} = \kappa_t + \sum_x \varepsilon_{x,t}^{(g)} \Rightarrow \sum_x \ln(m_{x,t}^{(g)}) = \kappa_t$

$\kappa_t^{(g)}$ vs. $\sum_x \log(m_{x,t}^{(g)})$ (Normalization Liu et al., 2019)

≈ 0, LLN



Motivation

- Lee-Carter is not the end of the story. It is rather the benchmark model that can be extended and modified into many dimensions.
- Modeling is always a trade-off between:
  - Tractability
  - Parsimony
  - Accuracy
- Unfortunately, there is no unified model that works best in all countries and in all populations. Instead, some of the variants work better in some countries, others work better in other countries.
- We will now briefly discuss some of the extensions and alternatives:
  - Trend correction
  - Alternative estimation approaches
  - Jump-off bias correction → not relevant for exam (skipped)
  - Alternative models
  - Multi-population models

Trend Correction

- Lee and Carter proposed to correct the estimated trend to fit the actual number of deaths, i.e., replace $\widehat{\kappa}_t^{(g)}$ by $\widetilde{\kappa}_t^{(g)}$ where $\widetilde{\kappa}_t^{(g)}$ is chosen such that

Lee-Carter estimator

$$\Rightarrow \quad \sum_{x \in \mathcal{X}} D_{x,t}^{(g)} = \sum_{x \in \mathcal{X}} E_{x,t}^{(g)} \exp\left(\widehat{\alpha}_x^{(g)} + \widehat{\beta}_x^{(g)} \widetilde{\kappa}_t^{(g)}\right).$$

- Alternative estimation methods (see next slide for illustration) incorporate this correction step.

Recall: Notation

Alternative Estimation: Brouhns et al. (2002)

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- Instead of SVD or OLS estimation, one can also apply a maximum likelihood approach to estimate the parameters $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$.
  - Avoids homoscedasticity of the errors occurring in OLS/SVD estimation.
  - Notice that the logarithm of the observed force of mortality is much more variable at older ages than at younger ages.
- Brouhns et al. (2002) model the number of deaths $D_{x,t}^{(g)}$ as Poisson-distributed random variables.
- Remember

$$\widehat{\mu}_{x,t}^{(g)} = m_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}.$$

$\Rightarrow$ $m_{x,t}^{(g)} \cdot E_{x,t}^{(g)} = D_{x,t}^{(g)}$

$\Downarrow$ holds in Expectation if we replace $m_{x,t}^{(g)}$ by $\mu_{x,t}^{(g)}$

- A reasonable model for $D_{x,t}^{(g)}$ would thus be

$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}(\mu_{x,t}^{(g)} E_{x,t}^{(g)}).$$

$\mathbb{E}[D_{x,t}^{(g)}]$

Alternative Estimation: Brouhns et al. (2002)

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$\ln(\mu_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t$

Uncertainty is now captured by the Poisson distribution [illegible] E_{x,t}^{(g)}

- Consequently,

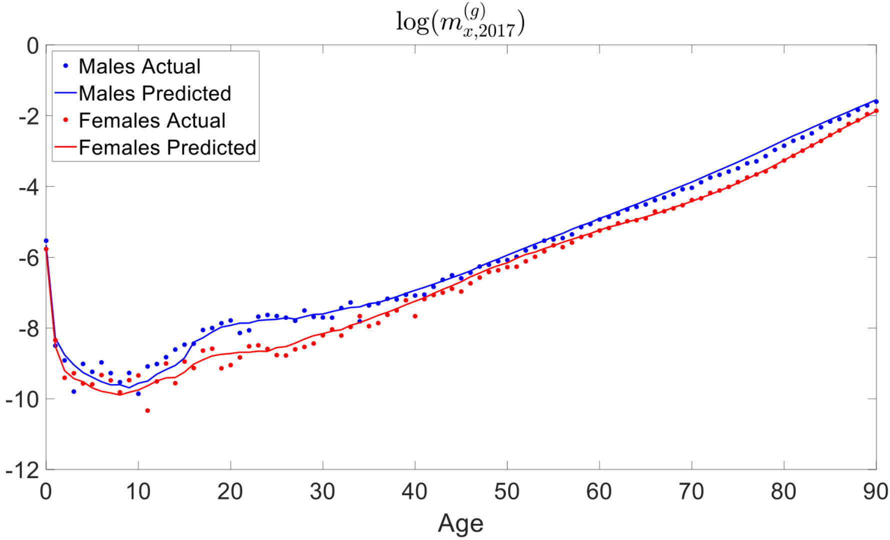
$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}\left(\mathrm{e}^{\alpha_x^{(g)}+\beta_x^{(g)}\kappa_t^{(g)}} E_{x,t}^{(g)}\right).$$

- Set up the log-likelihood function over all observations (x, t) and maximize it with respect to all $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$:

$$\ell(\alpha, \beta, \kappa) = \sum_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left[D_{x,t}^{(g)}\left(\alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)}\right) - E_{x,t}^{(g)}\mathrm{e}^{\alpha_x^{(g)}+\beta_x^{(g)}\kappa_t^{(g)}}\right].$$

- This maximization has again to be done iteratively. Normalizations such as $\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t\in\mathcal{T}} \kappa_t^{(g)} = 0$ are also needed again.
- This method is applied in the AG2022 model.

Jump-off Bias



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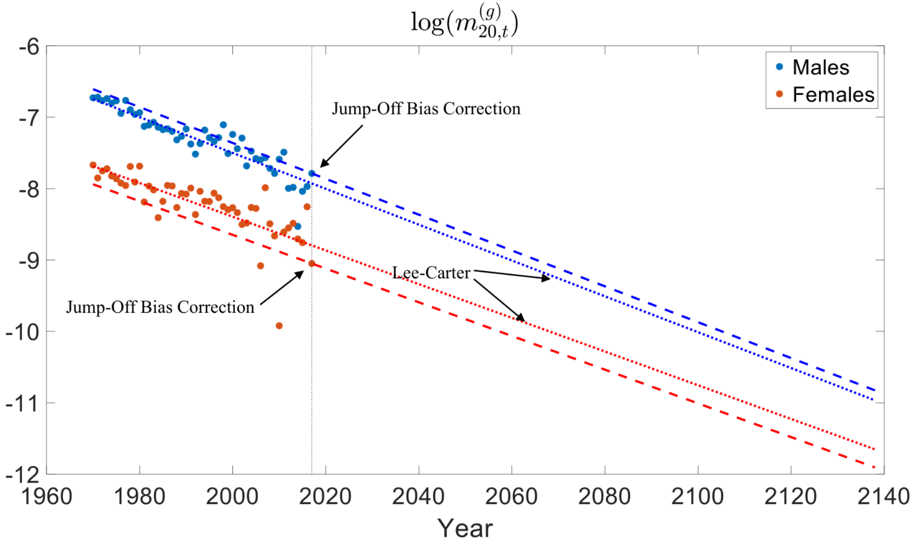
- Lee-Carter model: $\widehat{m}^{(g)}_{x,T+t_i} = \exp\left(\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i}\right)$, $i = 1, \ldots, N$.
- *Problem*: possibility of jump-off bias, i.e., $\widehat{m}^{(g)}_{x,T} \neq m^{(g)}_{x,T}$, where the latter is observable.
- To avoid this *jump-off bias* shift $\ln \widehat{m}^{(g)}_{x,T+t_i} = \widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i}$ by

$$\ln m^{(g)}_{x,T} - (\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_T)$$

to get an alternative forecast (correcting for jump-off bias):

$$\begin{aligned} \ln \widetilde{m}^{(g)}_{x,T+t_i} &= \widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i} + \ln m^{(g)}_{x,T} - (\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_T) \\ &= \ln m^{(g)}_{x,T} + \widehat{\beta}^{(g)}_x \left(\widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T\right). \end{aligned}$$

Jump-off Bias Correction



- Going from the forecast $\widehat{m}^{(g)}_{x,T+t_i} = \exp\left(\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i}\right)$, $i = 1, \ldots, N$ to

$$\ln \widetilde{m}^{(g)}_{x,T+t_i} = \ln m^{(g)}_{x,T} + \widehat{\beta}^{(g)}_x \left(\widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T\right)$$

  means replacing
  - $\widehat{\alpha}^{(g)}_x = \frac{1}{T}\sum_{t\in\mathcal{T}} \ln m^{(g)}_{x,t}$ by $\widetilde{\alpha}^{(g)}_x = \ln m^{(g)}_{x,T}$,
  - $\widehat{\kappa}^{(g)}_{T+t_i}$ by $\widetilde{\kappa}^{(g)}_{T+t_i} = \widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T$
- This way of avoiding the jump-off bias corresponds to the use of *reduction factors*

$$\widetilde{m}^{(g)}_{x,T+t_i} = m^{(g)}_{x,T} \cdot \mathrm{RF}^{(g)}_{x,t_i}$$

  with $\mathrm{RF}^{(g)}_{x,t_i} = \exp\left(\widehat{\beta}^{(g)}_x [\widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T]\right)$.

Many Alternative Models

- **Extensions of the Lee-Carter model**
  - Extra factor(s): $\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}$

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_{1,x}^{(g)} \kappa_{1,t}^{(g)} + \beta_{2,x}^{(g)} \kappa_{2,t}^{(g)} + \ldots + \varepsilon_{x,t}^{(g)}$$

  - Cohort effect:

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \gamma_x^{(g)} \iota_{t-x}^{(g)} + \varepsilon_{x,t}^{(g)}$$

- **Modeling alternative indicators**
  - Cairns-Blake-Dowd (CBD-)models, see Exercise 15:

$$\ln\left(\frac{q_{x,t}^{(g)}}{p_{x,t}^{(g)}}\right)$$

  - Oeppen (2008):

$$\ln\left(\frac{D_{x,t}^{(g)}}{h_t^{(g)}}\right), \qquad \text{with} \qquad h_t^{(g)} = \sqrt[X]{\prod_{x=1}^{X} D_{x,t}^{(g)}}$$

Interpretation of the Time Trend

Model Performance in Different Countries

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Average root mean square error ($\overline{RMSE}$) over forecast horizon h of the forecast life expectancy at birth for the period 1990 to 2014, with the best $RMSE$ value per country in bold and preferred set of models (SP) for 18 countries, females and males

Females							Males						
Country	**M**	**Q**	**D**	**L**	**E**	**SP**	**Country**	**M**	**Q**	**D**	**L**	**E**	**SP**
DEU–E	1.26	1.25	1.17	1.13	**1.01**	M, Q, D, L, E	DEU–E	2.35	2.35	2.22	2.11	**1.93**	E
DNK	1.17	1.15	**0.76**	0.91	0.96	D	DNK	2.16	2.16	2.08	2.03	**1.88**	E
IRL	1.08	1.07	0.87	0.86	**0.61**	E	IRL	2.13	2.12	2.09	1.90	**1.67**	E
PRT	0.77	0.75	0.45	**0.35**	1.10	L	NLD	1.75	1.75	1.74	1.68	**1.47**	E
JPN	**0.67**	0.72	1.88	1.59	1.57	M	NOR	1.71	1.71	1.66	1.62	**1.43**	E
UK	0.65	0.64	**0.38**	0.47	0.40	D	PRT	1.50	1.49	1.11	0.78	**0.44**	E
NOR	0.50	0.48	**0.24**	0.28	0.26	D	ITA	1.28	1.27	0.96	0.92	**0.52**	E
NLD	0.50	0.49	0.56	0.51	**0.49**	M, Q, D, L, E	ESP	1.25	1.23	0.81	0.78	**0.52**	E
AUT	0.47	0.46	0.36	**0.33**	0.33	M, Q, D, L, E	UK	1.23	1.22	0.93	0.98	**0.88**	D, E
USA	**0.43**	0.44	0.66	0.57	0.60	M, Q	CHE	1.20	1.19	0.79	0.89	**0.78**	D, E
ITA	0.31	**0.30**	0.52	0.41	0.76	Q	AUT	1.11	1.11	0.85	0.83	**0.62**	E
AUS	0.27	**0.26**	0.38	0.30	0.40	M, Q, L	SWE	1.08	1.08	0.93	0.95	**0.81**	E
ESP	0.24	**0.23**	0.33	0.34	0.87	Q	FRA	1.03	1.02	**0.53**	0.66	0.65	D
FRA	**0.23**	0.23	0.69	0.52	0.59	M	DEU–W	0.92	0.91	0.67	0.66	**0.42**	E
FIN	**0.22**	0.22	0.84	0.54	0.46	M, Q	AUS	0.89	0.86	0.52	0.55	**0.48**	E
DEU–W	**0.21**	0.21	0.64	0.49	0.65	M	FIN	0.87	0.86	**0.42**	0.56	0.47	D
CHE	**0.21**	0.22	0.82	0.59	0.58	M	JPN	**0.59**	0.62	1.59	1.23	1.25	M
SWE	**0.17**	0.18	0.54	0.38	0.38	M	USA	0.49	0.48	0.27	0.28	**0.24**	E
Mean	0.52	**0.52**	0.67	0.59	0.67		Mean	1.31	1.30	1.12	1.08	**0.92**	

M.-P. Bergeron-Boucher, S. Kjærgaard, J. Oeppen, J.W. Vaupel (2019)

Multi Population Models

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$$\ln(m_{t,x}^{(g)}) = \underbrace{A_x^{(g)} + B_x^{(g)} K_t^{(g)}}_{\text{European effect}} + \underbrace{\alpha_x^{(g)} + B_x^{(g)} \kappa_t^{(g)}}_{\text{Country-specific deviation between European effect and our population}} \; \varepsilon_{x,t}^{(g)}$$

- **Li and Lee (2005)**
  - Different populations do not live in isolation. Instead, there is a lot of interaction.
  - Therefore, it seems implausible that the mortalities of similar populations will diverge in the long run.
  - Similar populations have a common (non-stationary) time trend, while the difference between each population's time trend and the common time trend is likely stationary.
  - Traditional estimation to be done in multiple steps.

$$\mu_{x,t}^{(g)} = \mu_{x,t}^{(g),EU} \cdot \mu_{x,t}^{(g),dev.}$$

- **Application: AG2022-Model**
  - Similar populations determined based on GDP per capita.

Table of Contents

1 Introduction

2 Relevance of Macro Longevity Risk
- First Pillar: AOW
- Second Pillar: Pension Funds

3 Modeling Mortality

4 Benchmark Model
- The Lee-Carter Model
- Alternative Estimation
- Some Applications and Extensions

5 The AG2022 Model and COVID-19
- Model and Projections
- Closure of the Life Table

6 Model Risk: A Very Brief Introduction

Lecture Learning Goals – Lecture 12

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The lecture learning goals contribute to course goals [CG 4] and [CG 5]:
After successful completion of the course, the student will be able to . . .

- CG 4 compute software-based simulations, statistics, and visualizations of micro- and macro longevity risk as well as interest rate risk models based on real-world data.
- CG 5 analyze the effect of pandemics on macro longevity risk using the model of the Dutch Actuarial Association.

After successful completion of this lecture, the student will be able to . . .

- LG 1 critically evaluate the three layers of the model of the Dutch Actuarial Association. [CG 5].
- LG 2 implement the model of the Dutch Actuarial Association in a computer system and simulate the model forward [CG 4].
- LG 3 interpret the effect of the COVID-19 pandemic in the historical data and as a model outcome on life expectancy [CG 5].

General Structure

- So far, we have dealt with the Lee & Carter (1992) model and some of its variants. We now look into the AG2022 Model and COVID-19.
- The AG2022 Model consists of three layers (a so-called three-layer Lee-Li model):
  - A layer for the European population. → Lee Carter
  - A correction layer for the Dutch population. → deviation from the EU trend
  - An extra layer for COVID-19.
- The first two layers are as in AG2020.

Fixed Age effect Europe | Dynamic Age effect Europe | Fixed Age effect Deviation NL | Dynamic Age effect Deviation NL | Dynamic Age effect COVID in NL

$$\ln(\mu_{x,t}^g) = A_x^g + B_x^g K_t^g + \alpha_x^g + \beta_x^g \kappa_t^g + \mathfrak{B}_x^g \mathfrak{K}_t^g$$

Time effect Europe | Time effect Deviation NL | Time effect COVID in NL

Without Covid the model is identical to AG-2020

$$E[D_{x,t}^{(g)}] = \mu_{x,t}^{(g)} \cdot E_{x,t}^{(g)}$$

$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}(\mu_{x,t}^{(g)} \cdot E_{x,t}^{(g)})$$

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- The AG2022 directly builds upon AG2020.
- Surrounding countries selected based upon comparable welfare level.
- AG2020 re-estimated with data EU 2019 added (but data 2020 & 2021 not included). before Covid
  - EU data: $D_{x,t}^{(g),EU}$, $E_{x,t}^{(g),EU}$, $t = 1970, \dots, 2019$, $x = 0, \dots, 90$
  - Dutch data: $D_{x,t}^{(g),NL}$, $E_{x,t}^{(g),NL}$, $t = 1970, \dots, 2019$, $x = 0, \dots, 90$
- AG2020 is basis for long-term projections.
- The third layer is due to COVID-19 and is new. It is calibrated using data from 2020 and 2021, 2022, 2023

AG2020 Model Re-estimated, $t \leq 2019$

- Number of deaths ($D_{x,t}^{(g)}$) modeled assuming a Poisson-distribution with expectation equal to exposure ($E_{x,t}^{(g)}$) times force of mortality ($\mu_{x,t}^{(g),pre-covid}$):

$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}\left(E_{x,t}^{(g)} \mu_{x,t}^{(g),pre-covid}\right)$$

- The long-term mortality trend is estimated based on the European reference group:

$$\ln \mu_{x,t}^{(g),pre-covid,EU} = A_x^{(g)} + B_x^{(g)} K_t^{(g)}$$

- For the Netherlands mortality is estimated conditional on the force of mortality of the European reference group:

$$\ln \mu_{x,t}^{(g),pre-covid} = A_x^{(g)} + B_x^{(g)} K_t^{(g)} + \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)}$$

- All parameters are estimated by maximizing the corresponding likelihood functions.

AG2020 Model Re-estimated, $t \leq 2019$

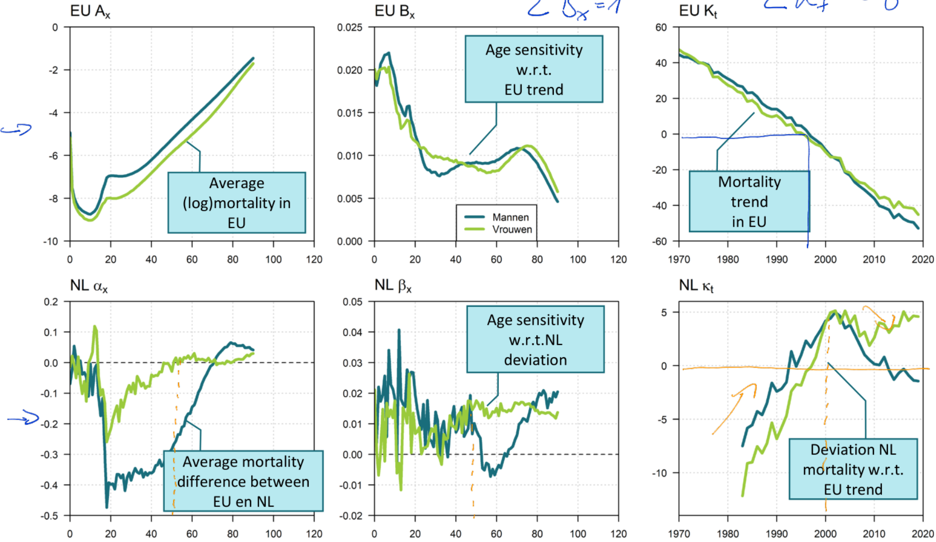
$$\ln \mu_{x,t}^{(g),\,pre\text{-}covid} = \underbrace{A_x^{(g)} + B_x^{(g)} K_t^{(g)}}_{\ln\left(\mu_{x,t}^{(g),\,pre\text{-}covid,\,EO}\right)} + \underbrace{\alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)}}_{\ln\left(\mu_{x,t}^{(g),\,pre\text{-}covid,\,dev}\right)}$$

$$= \ln\left(\mu_{x,t}^{(g),\,pre\text{-}covid,\,EO} \cdot \mu_{x,t}^{(g),\,pre\text{-}covid,\,dev}\right)$$

$$\Rightarrow \boxed{\mu_{x,t}^{(g),\,pre\text{-}covid} = \mu_{x,t}^{(g),\,pre\text{-}covid,\,EO} \cdot \mu_{x,t}^{(g),\,pre\text{-}covid,\,dev}}$$

Estimation

Same normalizations as in Lee-Carter $\sum B_x = 1$ $\sum K_t^{(g)} = 0$



Intuition

Let's consider women of age $x \geq 50$, $t \geq 2000$

$\alpha_x > 0$ $\quad$ $\kappa_t > 0$

$\beta_x > 0$

$$\mu_{x,t}^{(f),dev} = e^{\alpha_x^{(f)} + \beta_x^{(f)} \cdot \kappa_t^{(S)}} > 1$$

$$\Rightarrow \mu_{x,t}^{(f)} = \mu_{x,t}^{(f),EU} \cdot \mu_{x,t}^{(f),dev} > \mu_{x,t}^{(f),EU}$$

$\Rightarrow$ Mortality for women of age $x \geq 50$ is higher in NL than in the European reference group

AG2020 Model – Simulation

- The European period effects, $K_t^{(g)}$, are assumed to follow a random walk with drift (as in Lee-Carter):

$$\Delta K_t^{(g)} = \theta^{(g)} + \varepsilon_t^{(g)}$$

- The Dutch period effects, $\kappa_t^{(g)}$, are assumed to follow a first-order autoregressive process with constant:

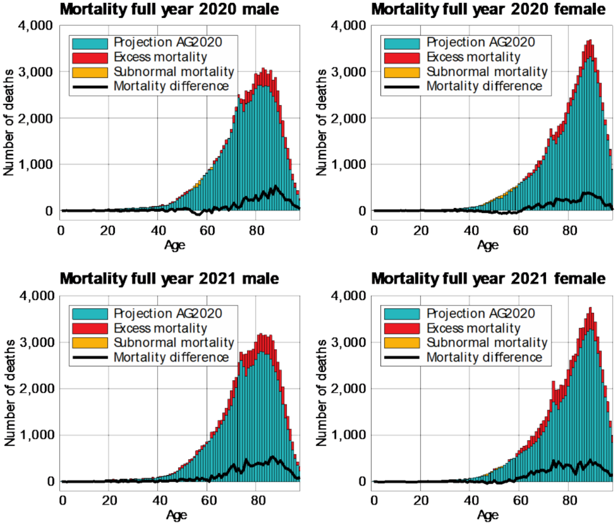
$$\Delta \kappa_t^{(g)} = a^{(g)} \kappa_{t-1}^{(g)} - c^{(g)} + \delta_t^{(g)}$$

$$= \omega^{(g)} \left(\tilde{\theta}^{(g)} - \kappa_{t-1}^{(g)} \right) + \delta_t^{(g)}$$

- The error terms are assumed to follow a multivariate normal distribution with mean vector 0 and covariance matrix C.
- This combination of assumptions results in coherent projections:
  - In the long run the difference between the Dutch and the European death probabilities converges to zero, but:
  - In the short run the Dutch and European death probabilities might deviate.

AR(1)-Process

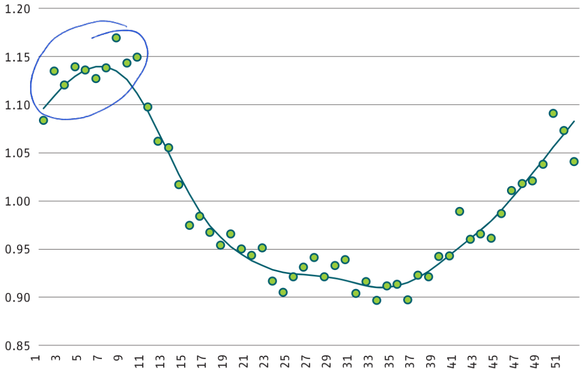
Third Layer: COVID-19



- Clear impacts of COVID-19 in both 2020 and 2021.
- Age effect in AG2022 differs from age effect in AG2020.

Third Layer: COVID-19

- Data from 2020 and 2021 cannot be used for normal update.
- Change to weekly data:
  - Customized data needed, plus interpolation and extrapolation.
  - Seasonal effect.
  - New age- and time effect.

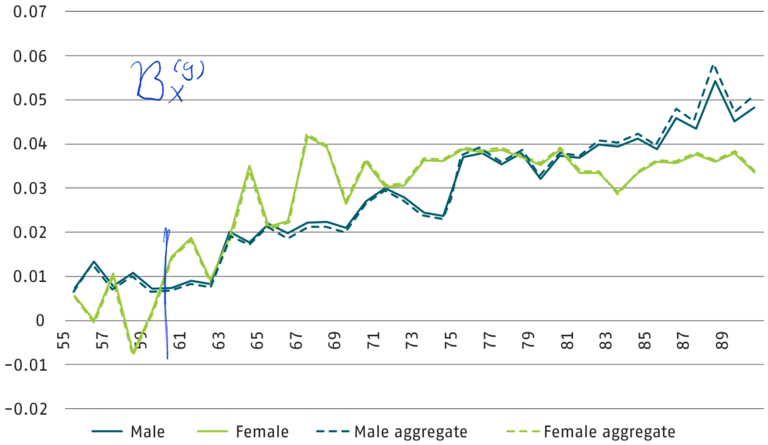


- To describe the deviation in mortality in 2020 and 2021 relative to what is expected on the basis of data before 2020, we introduce a new age effect $\mathfrak{B}_x^{(g)}$ and new (week-based) timeseries $\mathfrak{K}_{w,2020}^{(g)}$ for 2020 and $\mathfrak{K}_{w,2021}^{(g)}$ for 2021.
- The seasonal effect determined earlier is also added.
- Three factors:
  - the pre-pandemic estimate for a given year (an AG2020 model update with datapoints until 2019),
  - a seasonal effect for a given week,
  - a new factor representing the impact of the altered circumstances since the pandemic:

seasonal effect — dynamic age effect in a certain week

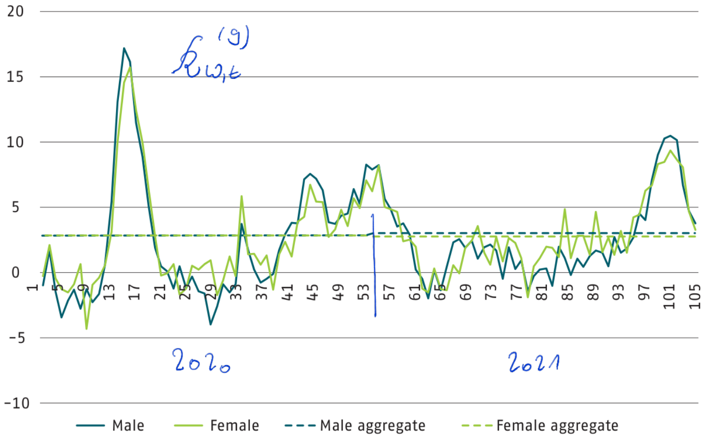
$$\ln \mu_{x,t}^{(g)} = \ln \mu_{x,t}^{(g),pre-covid} + \ln \phi_{w,t} + \mathfrak{B}_x^{(g)} \mathfrak{K}_{w,t}^{(g)}$$

- The time effect of COVID-19 $\mathfrak{K}_{w,t}^{(g)}$ varies significantly across weeks.



- Using weekly data allows the estimation of age effects based on more than 2 (annual) data points.
- No effect until age 55, constant from age 90.

Third Layer: Time Effect



- Time series for difference in mortality (whether or not due to COVID) relative to AG2020 model (with update 2019 data).
- Aggregating week effects $\mathfrak{K}_{w,t}^{(g)}$ gives estimate of the impact $\mathfrak{X}_t^{(g)}$ for the whole year (dashed lines).

Third Layer: Projection Model

- Estimating the impact for a whole year is the starting point for projecting future years.
- For the course of the projection further assumptions are needed.
- Choice CSO: disappearing (exponentially), i.e.,

$$\mathfrak{X}_t^{(g)} = \mathfrak{X}_{2021}^{(g)} \eta^{t-2021}, \qquad t \geq 2021$$

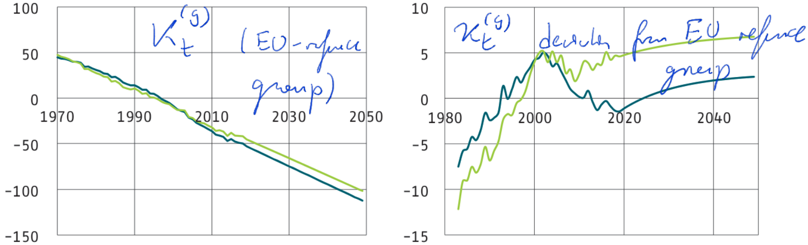
AG-2022

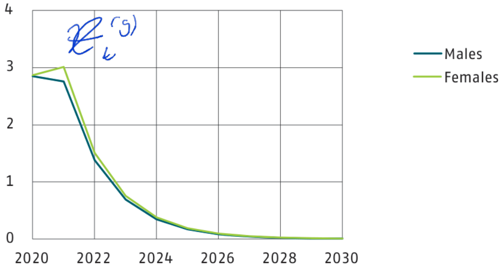
with $\eta = 1/2$, which implies that the half-life of the impact equals one year.

→ $\eta = 3/4$ in 2024

- This assumption determines the 'best estimates' for all future values of the time series in the model.

Third Layer: Projection Model



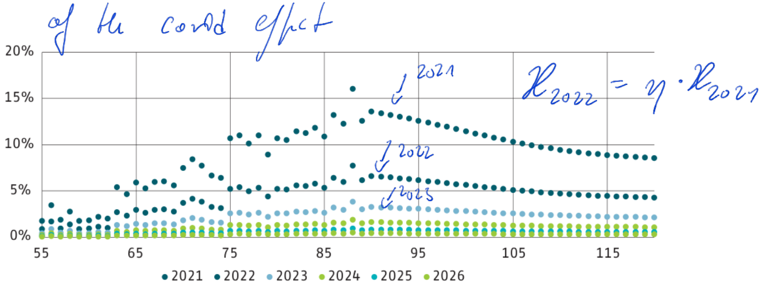


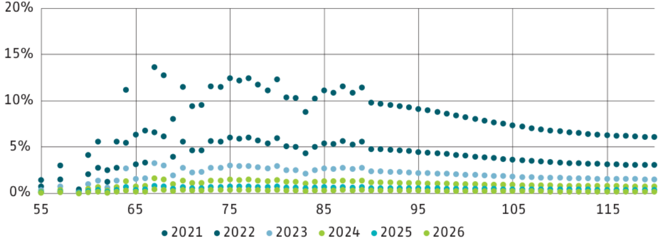
$$\Delta K_t^{(g)} = \theta^{(g)} + \varepsilon_t^{(g)}$$

$$\Delta \kappa_t^{(g)} = a^{(g)}\left(\tilde{\theta}^{(g)} - \kappa_t^{(g)}\right) + \delta_t^{(g)}$$

$$\mathcal{K}_t^{(g)} = \mathcal{K}_{t-1}^{(g)} \cdot \eta$$

Third Layer: Projection Model

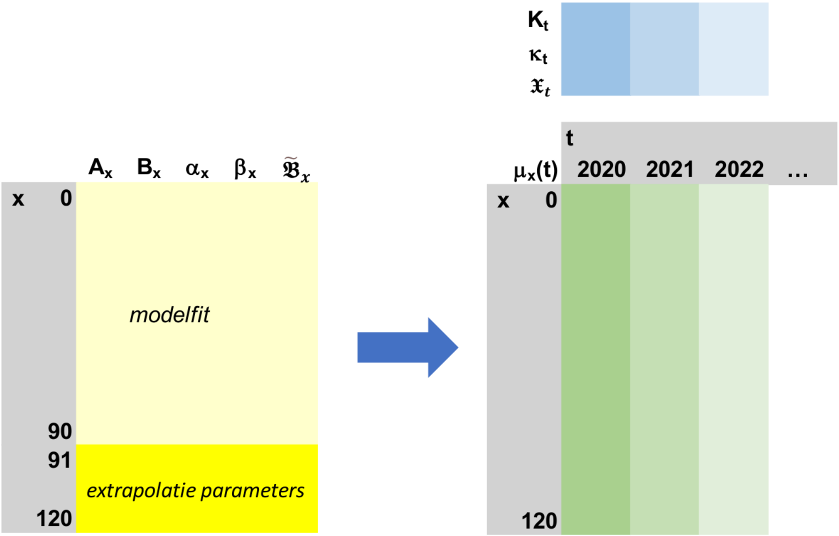




- *Problem:* The AG2022 model uses mortality data for $x = 0, \ldots, 90$. If $x \geq 91$, the data becomes rather noisy, because the population older than 90 is small.
- To deal with this issue, the AG2020 model used a popular method to extrapolate mortality rates for older ages – the *Kannisto closing method*.
- For the ages $x \in \{91, \ldots, 120\}$, $\widehat{\mu}^{(g)}_{x,T+t}$ is determined as

$$\widehat{\mu}^{(g)}_{x,T+t} = L\Big(\sum_{k=80}^{90} w_k(x) L^{-1}(\widehat{\mu}^{(g)}_{k,T+t}) \Big),$$

where $L(z) = \frac{1}{1+\mathrm{e}^{-z}}$ and $L^{-1}(z) = \ln\left(\frac{z}{1-z}\right)$ and some weight function w_k (see Assingment).



A_x
B_x
α_x
β_x
$\widetilde{\mathfrak{B}}_x$
x
0
modelfit
90
91
extrapolatie parameters
120
K_t
κ_t
$\mathfrak{X}_t$
t
$\mu_x(t)$
2020
2021
2022
...
x
0
120

Issues with Kannisto

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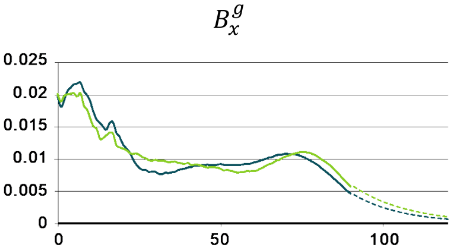
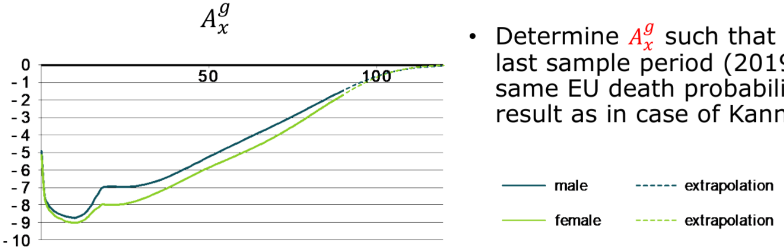
- The Kannisto method was standard in the AG-models and has been standard in many other forecasting models to deal with sparse data points in older cohorts.
- This method has some drawbacks:
  - Not all ages show improved mortality over time. For higher ages mortality rises monotonously to a positive and limit value that is known with certainty, $\lim_{t\to\infty} \widehat{q}^{(m)}_{x,T+t} = 0.6321$, $x \geq 101$. As a result, life expectancy also converges to a limit value known with certainty $\lim_{t\to\infty} e^{(m)}_{0,t} = 102.08$.
  - Because the limit values are known with certainty, uncertainty decreases (smaller confidence intervals) over time, while we expect increasing uncertainty (wider confidence intervals).
- The new method in AG2022 addresses this issue and closes the life table without knowing limit values with certainty.

Issues with Kannisto



- Extrapolate $\ln(B_x^g)$ linearly
- Interpretation:
  all ages benefit from decreasing trend in K_t



- Determine A_x^g such that in the last sample period (2019) the same EU death probabilities result as in case of Kannisto

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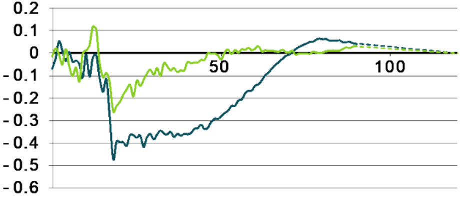
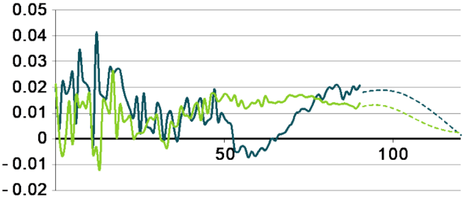
α_x^g



- Extrapolate α_x^g linearly to $\alpha_{120}^g = 0$
- Interpretation:
  for age 120 no difference between NL and EU.

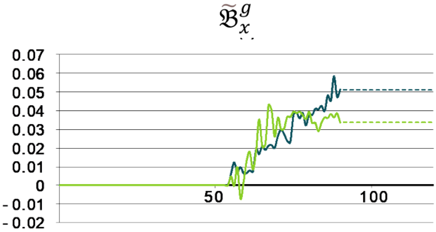
β_x^g



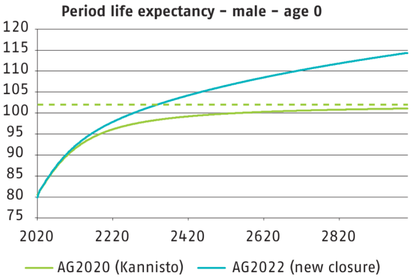
- Determine β_x^g such that in the last sample period (2019) the same NL death probabilities result as in case of Kannisto.

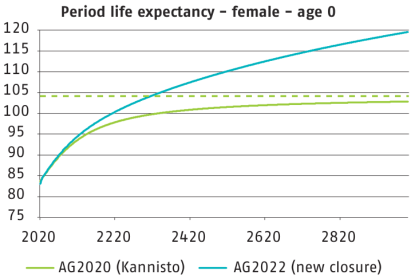
male ------ extrapolation
female ------ extrapolation

Closure of the Life Table



- Parameter COVID-term constant from age 90 on





- The new closing method does not have the undesirable characteristics, which we observe in the application of Kannisto per projection year.
- There is no turning point at a certain age: also, at higher ages the death probabilities keep decreasing.
- The life expectancy does not have a limit whose value is known in advance.
- The confidence intervals do not become smaller over time.

Closure of the Life Table

Table of Contents

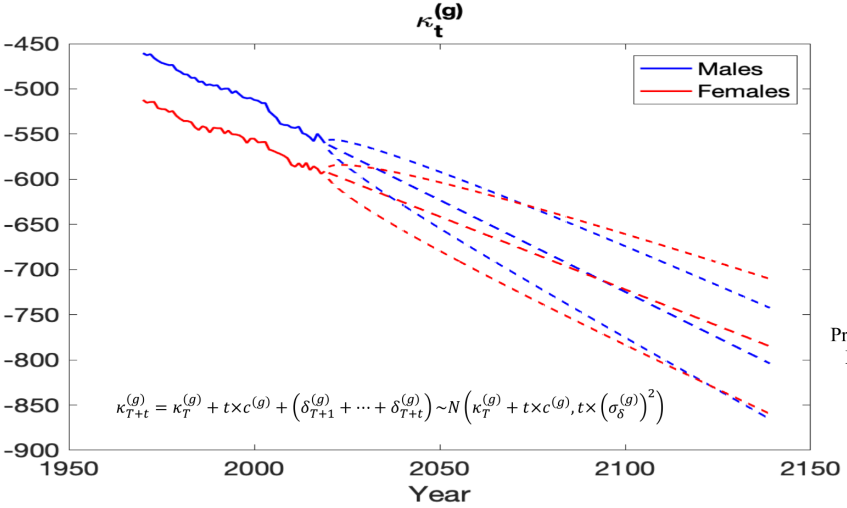
1. Introduction
2. Relevance of Macro Longevity Risk
   - First Pillar: AOW
   - Second Pillar: Pension Funds
3. Modeling Mortality
4. Benchmark Model
   - The Lee-Carter Model
   - Alternative Estimation
   - Some Applications and Extensions
5. The AG2022 Model and COVID-19
   - Model and Projections
   - Closure of the Life Table
6. Model Risk: A Very Brief Introduction

Three dimensions of model risk:

- **Parameter Risk**
  - Parameters ($\alpha_x^{(g)}$, $\beta_x^{(g)}$, $\kappa_t^{(g)}$, etc.) must be estimated, resulting in sampling inaccuracy.
- **Misspecification Risk**
  - Choice of model might be wrong.
- **Identification Risk**
  - Even if a model represents the past well and accurately, future might be different from past, i.e., past might not be representative for the future.

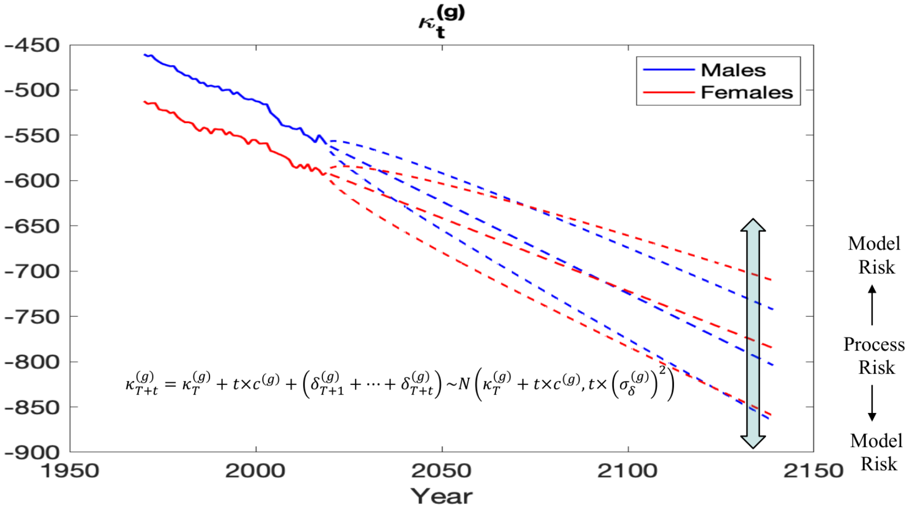
Process risk refers to the risk that the process evolves different than expected (e.g., generating too many paths outside the 95% confidence intervals).

Illustration Process Risk



Process Risk

Illustration Model Risk



Part IV

Pricing under all Types of Risk

Table of Contents

7 Setting

8 Illustrations
- No Risk
- Micro Longevity Risk
- Macro Longevity Risk
- Interest Rate Risk
- All Risks Combined

Ultimate Goal of the Lecture

- Now, we have gathered all the tools and techniques to price life insurance contracts and annuities in a realistic setting.
- We consider three different types of risk
  - Micro Longevity Risk (Part I): risk because (for given death probabilities) an individual's *remaining lifetime* is unknown.
  - Interest Rate Risk (Part II): risk because future *interest rates* are unknown.
  - Macro Longevity Risk (Part III): risk because *future death probabilities* are unknown.
- We study the pricing of life insurance contracts and annuities in various settings:
  - No risk (tedious but useful)
  - Micro longevity risk
  - Macro longevity risk
  - Interest rate risk
  - All risks combined

Recall: Three Sources of Risk

1. Micro Longevity Risk (i.i.d.): $N^{(g)}_{x+t,T+t} \sim \mathcal{B}(N^{(g)}_{x,T},\ {}_t p^{(g)}_{x,T})$

$$N^{(g)}_{x+t,T+t} = \underbrace{N^{(g)}_{x,T}\,{}_t p^{(g)}_{x,T}}_{\text{best estimate}} + \underbrace{E^{(g)}_{x+t,T+t}}_{\text{forecast error}}$$

2. Macro Longevity Risk (Lee-Carter): $p^{(g)}_{x,T+t} = \mathrm{e}^{-m^{(g)}_{x,T+t}}$, $\ln(m^{(g)}_{x,t}) = \alpha^{(g)}_x + \beta^{(g)}_x \kappa^{(g)}_t + \varepsilon^{(g)}_{x,t}$

$$\kappa^{(g)}_{T+t} = \underbrace{\kappa^{(g)}_T + t \cdot c^{(g)}}_{\text{best estimate}} + \underbrace{\delta^{(g)}_{T+1} + \cdots + \delta^{(g)}_{T+t}}_{\text{forecast error}}$$

3. Interest Rate Risk (Vasicek): $r_{t+1} = \mu + \theta r_t + \sigma \varepsilon_{t+1}$, $R_t(t+1) = r_t$

$$r_{T+t} = \underbrace{\mu \sum_{i=1}^{t-1} \theta^i + \theta^t r_T}_{\text{best estimate}} + \underbrace{\sigma \sum_{i=1}^{t} \theta^{t-i} \varepsilon_{T+i}}_{\text{forecast error}}$$

Recall: Three Sources of Risk

Setting

- We only consider a single group g (g suppressed from now on) at time t.
- Members of this group belong to a cohort (x, t). The number of individuals belonging to cohort (x, t) is given by $N_{x,t}$.
- All individuals have bought an immediate single life annuity from a fund at time t. This annuity promises to pay off 1 unit per period, starting in the next period $t + 1$.

  $\rightarrow$ This assumption is not realistic but helps us come up with closed-form solutions to illustrate the impact of the various risk sources.
- The fund invests the received payments in assets, in order to be able to pay off the promised amounts of the annuities.

- The annuity prices are based on "best estimate" cohort life table at time t, explicitly indicated by $BE(t)$, i.e.,

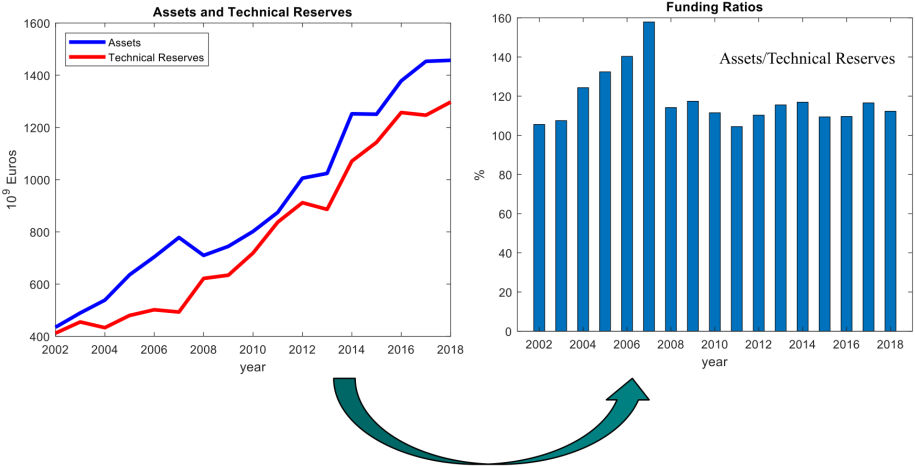
$$a_{x,t}^{BE(t)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{BE(t)} \frac{1}{\big(1 + R_t(t+\tau)\big)^{\tau}}.$$

- The pension fund's total liabilities $L_t^{BE(t)}$ are given by

$$L_t^{BE(t)} = \sum_{x \in \mathcal{X}} N_{x,t} a_{x,t}^{BE(t)}.$$

- At time t the fund's total assets are denoted by A_t.

Recall: Funding Ratio



Setting: Funding Ratio

- Thus, at time t, the fund's *funding ratio* (FR_t) is defined as

$$FR_t^{BE(t)} = \frac{A_t}{L_t^{BE(t)}}.$$

- The funding ratio does obviously depend on all three sources of risk under consideration.
- We ask the question: What can we say about FR_{t+1}, considering
  - No risk
  - Micro longevity risk
  - Macro longevity risk
  - Interest rate risk
  - All risks combined

Table of Contents

7 Setting

8 Illustrations
- No Risk
- Micro Longevity Risk
- Macro Longevity Risk
- Interest Rate Risk
- All Risks Combined

No Risk: Assumptions

We first consider the case without risks under the following assumptions:

1. The number of survivors equals the number of expected survivors (the "best estimate"), i.e.,

$$N_{x+1,t+1} = N_{x,t} p_{x,t}^{BE(t)}.$$

2. The "best estimate" cohort life table of time $t+1$ follows from the "best estimate" cohort life table of time t (by excluding the column corresponding to time t), i.e.,

$$p_{x,t+1+\tau}^{BE(t+1)} = p_{x,t+1+\tau}^{BE(t)}.$$

3. Financial assets generate known returns $r_{t+\tau}$ between periods $t+\tau$ and $t+\tau+1$ with

$$r_{t+\tau} = R_{t+\tau}(t+\tau+1).$$

- It's easy to check that without financial risk the following relation holds

$$\big(1 + R_t(t+\tau)\big)^{\tau} = \prod_{j=0}^{\tau-1}(1 + r_{j+\tau}).$$

- *Proof*:

- Consequently,

$$a_{x,t}^{BE(t)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{BE(t)} \prod_{j=0}^{\tau-1} \frac{1}{1 + r_{j+\tau}}.$$

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- We can now derive the following recursive relationship:

$$
\begin{aligned}
a_{x,t}^{BE(t)} &= \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{BE(t)} \prod_{j=0}^{\tau-1} \frac{1}{1+r_{j+\tau}} \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + \sum_{\tau=2}^{\infty} {}_{\tau-1}p_{x+1,t+1}^{BE(t)} \prod_{j=1}^{\tau-1} \frac{1}{1+r_{j+\tau}}\right) \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t)} \prod_{j=0}^{\tau-1} \frac{1}{1+r_{j+1+\tau}}\right) \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t)} \frac{1}{1+R_{t+1}(t+1+\tau)}\right) \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + a_{x+1,t+1}^{BE(t+1)}\right)
\end{aligned}
$$

- Consequently,

$$a_{x,t}^{BE(t)} = p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + a_{x+1,t+1}^{BE(t+1)}\right)$$

$$a_{x+1,t+1}^{BE(t+1)} = (1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1.$$

- Therefore, the liabilities at $t+1$ are now given by

$$\begin{aligned} L_{t+1}^{BE(t+1)} &= \sum_{x \in \mathcal{X}} N_{x+1,t+1} a_{x+1,t+1}^{BE(t+1)} \\ &= \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)} \left((1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1 \right) \\ &= (1+r_t) L_t^{BE(t)} - \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}. \end{aligned}$$

- The pension fund invests its assets at the capital market rate yielding a return r_t. Therefore, $\widetilde{A}_{t+1} = A_t(1 + r_t)$.
- Using these assets, the fund pays off the first unit of the annuities to all surviving members of the fund at time $t + 1$. The total payoff equals

$$\sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}.$$

- Thus, the resulting value of the assets at time $t + 1$ is given by

$$A_{t+1} = A_t(1 + r_t) - \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}.$$

- The funding ratio at time $t+1$ is thus

$$FR_{t+1}^{BE(t+1)} = \frac{A_{t+1}}{L_{t+1}^{BE(t+1)}} = \frac{A_t(1+r_t) - \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}}{L_t^{BE(t)}(1+r_t) - \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}}.$$

- Consequently, if
  - $FR_t^{BE(t)} < 1$: the funding ratio declines over time.
  - $FR_t^{BE(t)} = 1$: the funding ratio stays stable over time.
  - $FR_t^{BE(t)} > 1$: the funding ratio increases over time.

Micro Longevity Risk: Assumptions

- We adapt the following no longevity risk assumption (Assumption 1, Slide 156):

$$N_{x+1,t+1} = N_{x,t} p_{x,t}^{BE(t)}.$$

- Instead, we assume that survival probabilities are known in advance although the remaining lifetime is uncertain. We assume that the number of survivors after one period follows a binomial distribution:

$$N_{x+1,t+1}^{(g)} \sim \mathcal{B}(N_{x,t}^{(g)},\, p_{x,t}^{(g)}) \qquad \forall x \in \mathcal{X}.$$

- The "best estimate" cohort life table of time $t+1$ follows from the "best estimate" cohort life table of time t (by excluding the column corresponding to time t), i.e.,

$$p_{x,t+1+\tau}^{BE(t+1)} = p_{x,t+1+\tau}^{BE(t)}.$$

- For the moment, we stick to the assumption that financial assets generate known returns $r_{t+\tau}$ between periods $t+\tau$ and $t+\tau+1$ with

$$r_{t+\tau} = R_{t+\tau}(t+\tau+1).$$

- Under these assumptions, the recursive relationship for the price of an annuity contract remains valid, i.e.,

$$a_{x,t}^{BE(t)} = p_{x,t}^{BE(t)} \frac{1}{1+r_t}\left(1 + a_{x+1,t+1}^{BE(t+1)}\right),$$

$$a_{x+1,t+1}^{BE(t+1)} = (1+r_t)\frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1.$$

- We can now calculate the one-period-ahead assets, liabilities, and the funding ratio:
- Assets (same calculations as before):

$$A_{t+1} = A_t(1 + r_t) - \sum_{x \in \mathcal{X}} \underbrace{N_{x+1,t+1}}_{\neq N_{x,t} p_{x,t}^{BE(t)}}$$

- Liabilities (same calculations as before):

$$L_{t+1}^{BE(t+1)} = \sum_{x \in \mathcal{X}} N_{x+1,t+1} \left((1 + r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1 \right)$$

- Funding ratio:

$$FR_{t+1} = \frac{A_{t+1}}{L_{t+1}^{BE(t+1)}} = \frac{A_t(1 + r_t) - \sum_{x \in \mathcal{X}} N_{x+1,t+1}}{\sum_{x \in \mathcal{X}} N_{x+1,t+1} \left((1 + r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1 \right)}$$

- We illustrate micro longevity risk using the following stylized example.
- We consider one age group ($x = 65$) at one specific year (namely, $t = 2019$).
- We present results for males and females separately.
- We generate 10,000 scenarios.
- We consider $N_{65,2019} = 1{,}000$, $10{,}000$, and $50{,}000$.
- We assume $r_{t+\tau} = 0$ for all $\tau \geq 0$.
- For the life table at time t, we use the Lee-Carter best estimate, based on the sample of Dutch males and females 1970-2019.
  - Males: $q_{65,2019}^{BE(2019)} = 1.07\%$, $p_{65,2019}^{BE(2019)} = 98.93\%$, $a_{65,2019}^{BE(2019)} = 18.95$.
  - Females: $q_{65,2019}^{BE(2019)} = 0.76\%$, $p_{65,2019}^{BE(2019)} = 99.24\%$, $a_{65,2019}^{BE(2019)} = 21.96$.

- To simulate a sample from a binomial distribution with parameters n and p, you could make use of the command

  `binornd(n,p),`

  where $n = N_{65,2019}$ and $p = p_{65,2019}^{BE(2019)}$.

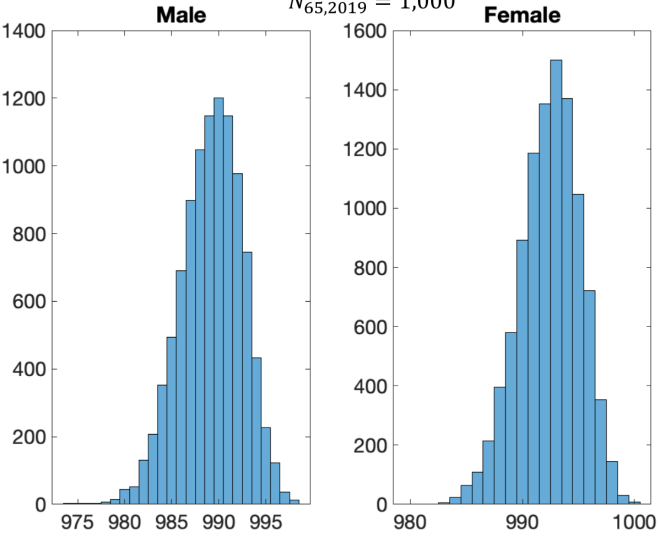
- To make a histogram of the data in the vector y (e.g., simulated funding ratio), you could make use of the command

  `histogram(y,edges),`

  where `edges` is a vector with the edges of the histogram columns.

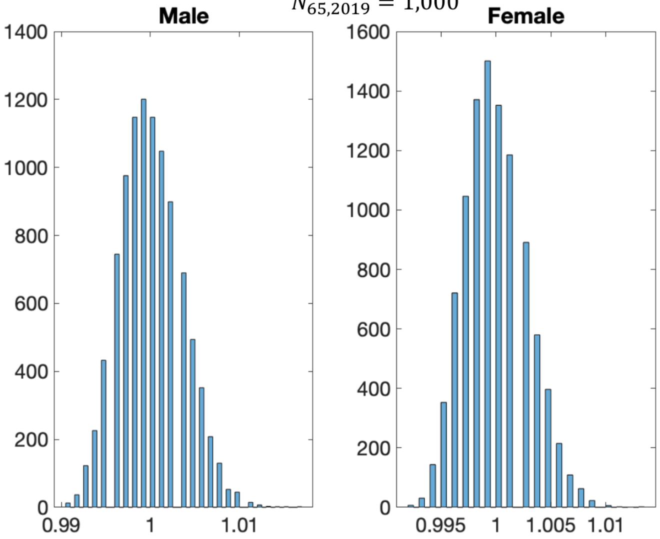
Histogram Survivors

$N_{65,2019} = 1{,}000$

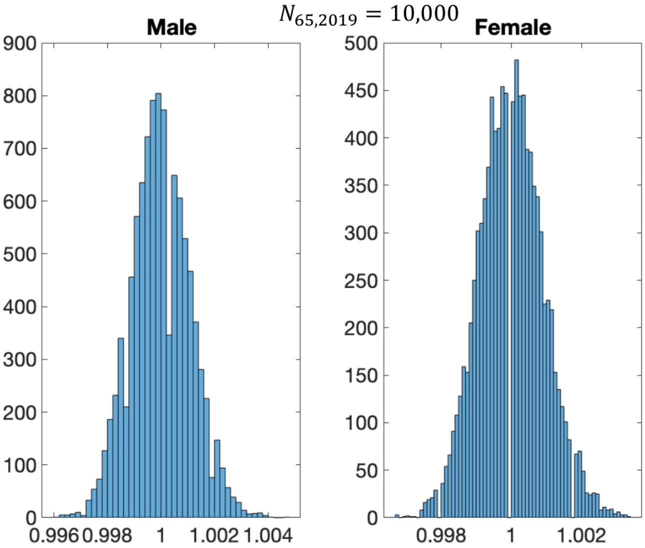


Histogram Funding Ratio

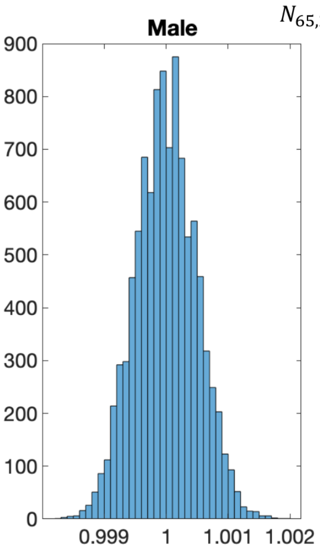
$N_{65,2019} = 1{,}000$



Histogram Funding Ratio



Histogram Funding Ratio



Macro Longevity Risk: Assumptions

- We adapt the following no macro longevity risk assumption (Assumption 2, Slide 156):

$$p_{x,t+1+\tau}^{BE(t+1)} = p_{x,t+1+\tau}^{BE(t)}$$

- Instead of assuming that the best estimate cohort life table of time $t+1$ follows from the best estimate cohort life table of time t, we follow a model-based approach.

- There are many ways in which Macro Longevity Risk can be quantified.

- We choose and illustrate one specific way, focusing on the Lee-Carter approach.

- Recall that the best estimate survival probabilities are constructed as follows:

$$p_{x,t+\tau}^{BE(t)} = \exp\big(-\widehat{m}_{x,t+\tau}^{BE(t)}\big),$$

- where

$$\widehat{m}_{x,t+\tau}^{BE(t)} = \exp\big(\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)}\widehat{\kappa}_{t+\tau}^{BE(t)}\big),$$
$$\widehat{\kappa}_{t+\tau}^{BE(t)} = \widehat{\kappa}_t^{(t)} + \tau\widehat{c}^{(t)}$$

- The superindex t in case of $\widehat{\alpha}_x^{(t)}, \widehat{\beta}_x^{(t)}, \widehat{c}^{(t)}, \widehat{\kappa}_t^{(t)}$ refers to the sample based on which these parameters have been estimated, namely, the sample with data from time $t_0 = t - T$ to $t_0 + T = t$.

Macro Longevity Risk: Simplifying Assumptions

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- To keep the setting simple (although the notation is already cumbersome), we make the following simplifying assumption:
- Macro longevity risk only due to change in $\widehat{\kappa}_{t+1}^{(t+1)}$ compared to the best estimate $\widehat{\kappa}_{t+1}^{BE(t)} = \widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)}$:

$$\widehat{\kappa}_{t+1}^{(t+1)} = \widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1}.$$

- We then (re-)estimate $\widehat{c}^{(t+1)}$, now using the sample with data from time $t_0 = t - T$ to $t_0 + T + 1 = t + 1$:

$$\widehat{c}^{(t)} = \frac{\widehat{\kappa}_t^{(t)} - \widehat{\kappa}_{t-T}^{(t)}}{T-1} \quad \longrightarrow \quad \widehat{c}^{(t+1)} = \frac{\widehat{\kappa}_{t+1}^{(t+1)} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T}$$

- No macro longevity risk due to a possible change in $\widehat{\alpha}_x^{(t+1)}$ or $\widehat{\beta}_x^{(t+1)}$.

Macro Longevity Risk: Next Period's Life Table

- Recall: $p_{x,t+\tau}^{BE(t)} = \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \widehat{\kappa}_{t+\tau}^{BE(t)}}}$
- Therefore,

$$
\begin{aligned}
p_{x,t+1+\tau}^{BE(t+1)} &= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t+1)} + \widehat{\beta}_x^{(t+1)} \widehat{\kappa}_{t+1+\tau}^{BE(t+1)}}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_{t+1}^{(t+1)} + \tau \widehat{c}^{(t+1)} \right)}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1} + \tau \frac{\widehat{\kappa}_{t+1}^{(t+1)} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T} \right)}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1} + \tau \frac{\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T} \right)}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \tau \frac{\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T} + \delta_{t+1} (1 + \frac{\tau}{T}) \right)}}
\end{aligned}
$$

Macro Longevity Risk: Assets and Liabilities

- Maintaining assumptions 1 and 3 from slide 156, we can now calculate the evolution of assets and liabilities.
- Assets (same calculations as before):

$$A_{t+1} = A_t(1+r_t) - \sum_{x\in\mathcal{X}} \underbrace{N_{x+1,t+1}}_{=N_{x,t}p_{x,t}}$$

$$= (1+r_t)\sum_{x\in\mathcal{X}} N_{x,t}a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}$$

- Liabilities (same calculations as before):

$$L_{t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x+1,t+1}a_{x+1,t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}a_{x+1,t+1}^{BE(t+1)}$$

where

$$a_{x+1,t+1}^{BE(t+1)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t+1)} \prod_{j=0}^{\tau-1} \frac{1}{1+r_{t+1+j}}.$$

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- Consequently, the funding ratio becomes

$$FR_{t+1} = \frac{(1+r_t)\sum_{x\in\mathcal{X}} N_{x,t}a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}}{\sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}a_{x+1,t+1}^{BE(t+1)}}.$$

- To illustrate the effect of macro longevity risk, we consider only one age group:

$$FR_{t+1} = \frac{(1+r_t)a_{x,t}^{BE(t)} - p_{x,t}}{p_{x,t}a_{x+1,t+1}^{BE(t+1)}}$$

- We illustrate macro longevity risk using the following stylized example.
- We consider one age group ($x = 65$) at one specific year (namely, $t = 2019$).
- We present results for males and females separately.
- We generate 10,000 scenarios.
- We assume $r_{t+\tau} = 0$ for all $\tau \geq 0$.
- We assume $\delta_{t+1} \sim \mathcal{N}\left(0, (\sigma_\delta^{(g)})^2\right)$ with $\sigma_\delta^{(m)} = 2.8653$, $\sigma_\delta^{(f)} = 3.4771$.
- For the life table at time t, we use the Lee-Carter best estimate, based on the sample of Dutch males and females 1970-2019.
  - Males: $q_{65,2019}^{BE(2019)} = 1.07\%$, $p_{65,2019}^{BE(2019)} = 98.93\%$, $a_{65,2019}^{BE(2019)} = 18.95$.
  - Females: $q_{65,2019}^{BE(2019)} = 0.76\%$, $p_{65,2019}^{BE(2019)} = 99.24\%$, $a_{65,2019}^{BE(2019)} = 21.96$.

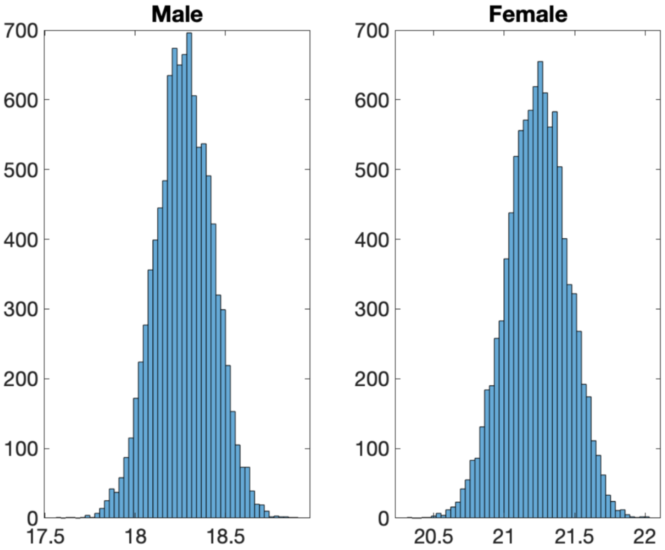
Implementation in Matlab

- To simulate a sample from a standard normal distribution, you could make use of the command

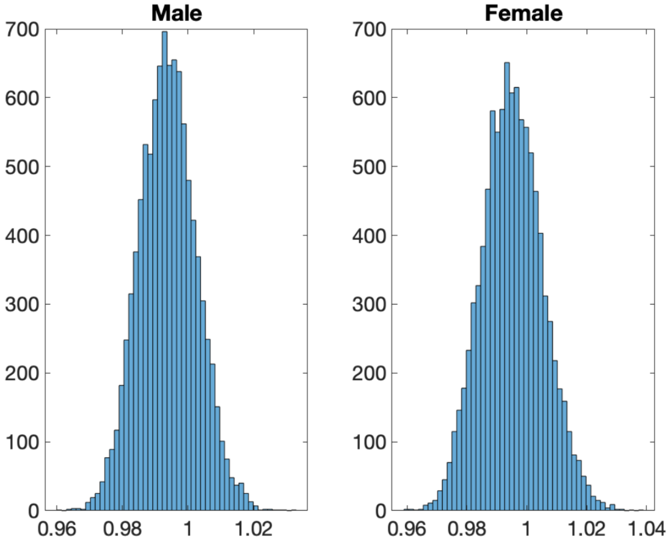
```
Z=randn(nrows,ncolumns).
```

- The matrix `Z` with dimension `nrows` $\times$ `ncolumns` then contains pseudo random numbers from a standard normal distribution.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, we can write $X = \mu + \sigma Z$, where $Z \sim \mathcal{N}(0, 1)$. This can be used to get a sample from a $\mathcal{N}(\mu, \sigma^2)$-distribution.

Histogram Annuity Factor



Histogram Funding Ratio



Interest Rate Risk: Assumptions

- We now adapt Assumption 3 on Slide 156 that returns are known in advance. We rather replace it by a term structure model for the evolution of interest rates.
- There are many ways to do this. As illustration, we shall make use of the Vasicek model, applied to term structure data provided by DNB (the Dutch Central Bank),

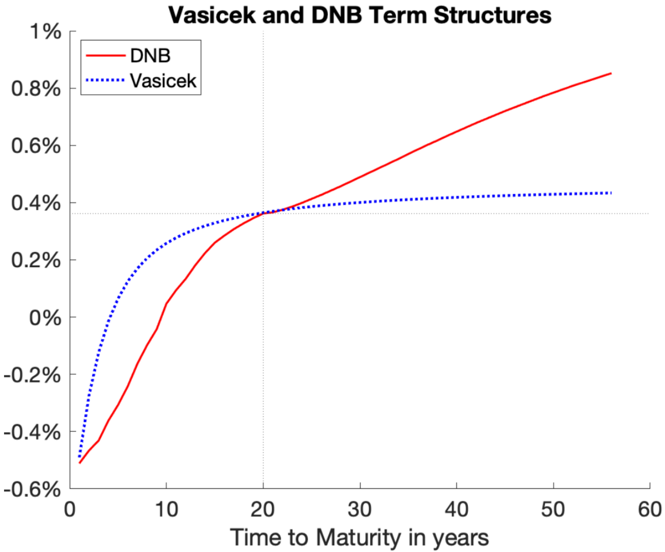
$$r_{t+1} = \mu + \theta r_t + \sigma \varepsilon_{t+1}, \qquad R_t(t+1) = r_t.$$

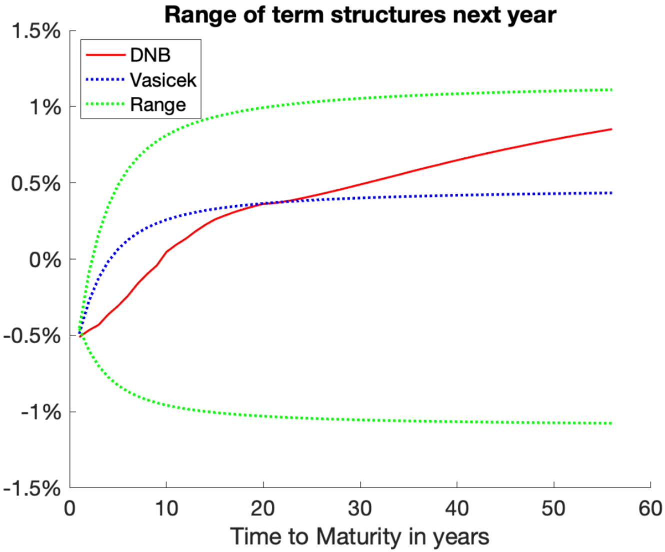
- Based on DNB-data from 2019, we find

$$r_{t+1} = 0.0018 + 0.5522 r_t + 0.0026 \varepsilon_{t+1}.$$

- In the lecture Valuation and Risk Management (QFAS master), we are going to study various other interest rate risk models and discuss the pros and cons of these models.

Interest Rate Risk: Fit to the Data




Range of term structures next year
DNB
Vasicek
Range
1.5%
1%
0.5%
0%
-0.5%
-1%
-1.5%
0
10
20
30
40
50
60
Time to Maturity in years

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- Maintaining assumptions 1 and 2 on slide 156, we can now calculate the evolution of assets and liabilities.
- Assets (same calculations as before):

$$\begin{aligned} A_{t+1} &= A_t(1+r_t) - \sum_{x\in\mathcal{X}} \underbrace{N_{x+1,t+1}}_{=N_{x,t}p_{x,t}} \\ &= (1+r_t)\sum_{x\in\mathcal{X}} N_{x,t}a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t} \end{aligned}$$

- Liabilities (same calculations as before):

$$L_{t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x+1,t+1}a_{x+1,t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}a_{x+1,t+1}^{BE(t+1)}$$

where

$$a_{x+1,t+1}^{BE(t+1)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t+1)} \frac{1}{\big(1+R_{t+1}(t+1+\tau)\big)^{\tau}}.$$

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- Consequently, the funding ratio becomes

$$FR_{t+1} = \frac{(1+r_t)\sum_{x\in\mathcal{X}} N_{x,t} a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t}}{\sum_{x\in\mathcal{X}} N_{x,t} p_{x,t} a_{x+1,t+1}^{BE(t+1)}}.$$

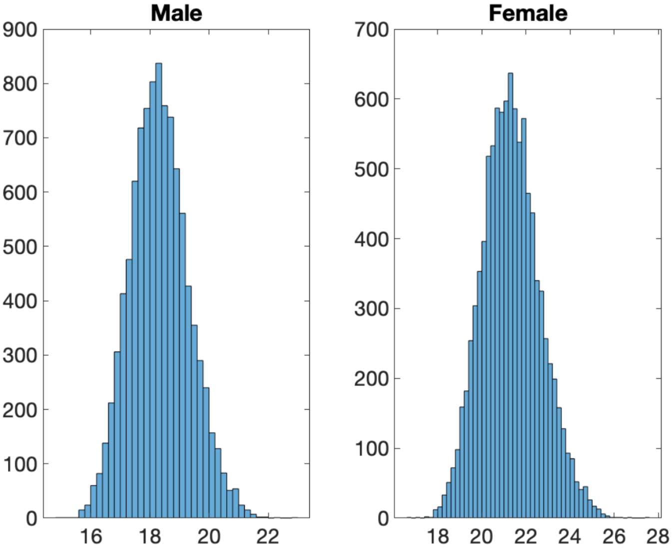
- To illustrate the effect of interest rate risk, we consider only one age group:

$$FR_{t+1} = \frac{(1+r_t) a_{x,t}^{BE(t)} - p_{x,t}}{p_{x,t} a_{x+1,t+1}^{BE(t+1)}}$$

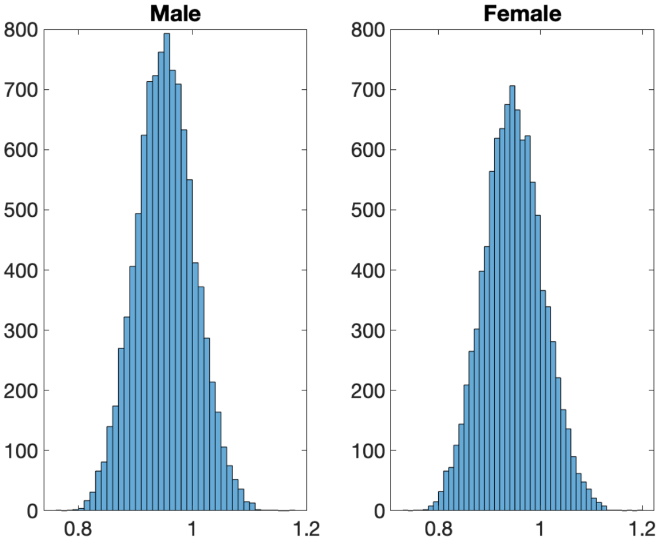
Illustration

- We illustrate interest rate risk using the following stylized example.
- We consider one age group ($x = 65$) at one specific year (namely, $t = 2019$).
- We present results for males and females separately.
- We generate 10,000 scenarios.
- We assume $r_{t+1} = 0.0018 + 0.5522 r_t + 0.0026 \varepsilon_{t+1}$ for all $\tau \geq 0$ and $r_{2019} = -0.51\%$.
- For the life table at time t, we use the Lee-Carter best estimate, based on the sample of Dutch males and females 1970-2019.
  - Males: $q_{65,2019}^{BE(2019)} = 1.07\%$, $p_{65,2019}^{BE(2019)} = 98.93\%$, $a_{65,2019}^{BE(2019)} = 18.95$.
  - Females: $q_{65,2019}^{BE(2019)} = 0.76\%$, $p_{65,2019}^{BE(2019)} = 99.24\%$, $a_{65,2019}^{BE(2019)} = 21.96$.

Histogram Annuity Factor



Histogram Funding Ratio

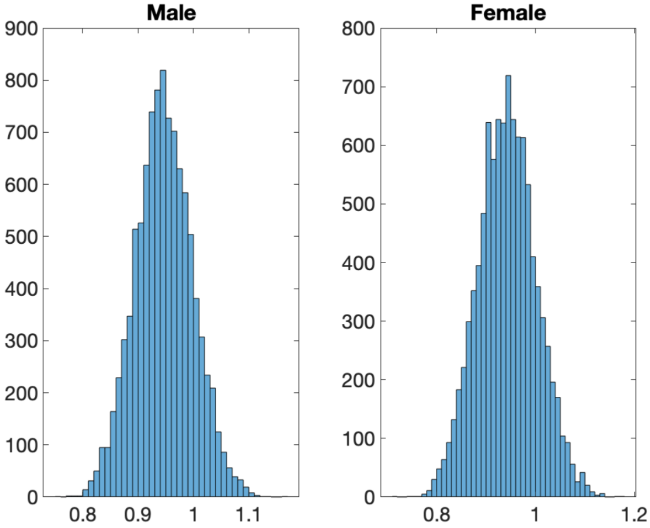


- We now switch on all types of risk as described in the previous sections.
- In addition, we add stock market risk and allow the pension fund to invest parts of its assets in risky stocks.
- We assume an annual volatility of $\sigma = 20\%$ and an expected annual rate of return of $\mu = 5\%$, where the return is normally distributed, i,e., $R_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$ (Markowitz model).
- Assuming that the fund invests a fraction π of its assets in the stock market, the assets evolve according to

$$A_{t+1} = A_t(1-\pi)(1+r_t) + A_t\pi(1+R_{t+1}) - \sum_{x\in\mathcal{X}} N_{x+1,t+1}$$

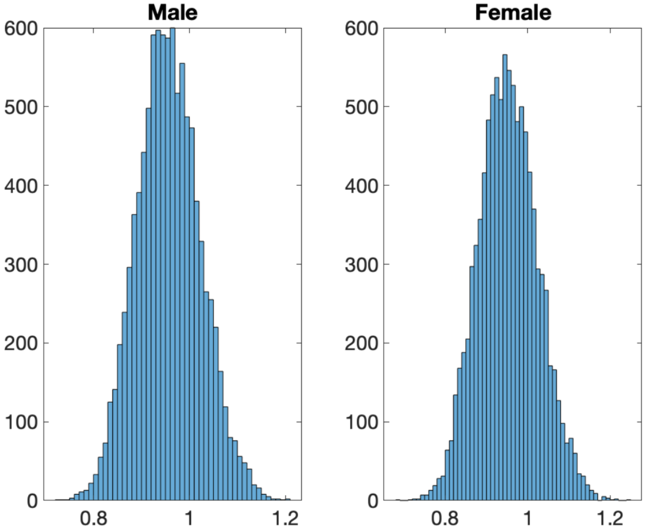
$$= A_t[1 + r_t + \pi(R_{t+1} - r_t)] - \sum_{x\in\mathcal{X}} N_{x+1,t+1}.$$

Histogram Funding Ratio



Investment Assets: $\pi = 0\%$ stock.

Histogram Funding Ratio



Investment Assets: $\pi = 20\%$ stock, with expected annual return $\mu = 5\%$ and annual volatility $\sigma = 20\%$.