General exam data			
Name subject	Life Insurance	Subject code	35V6A3-B-6
Lecturer	Hambel, Drost	ANR lecturer	51629
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Length of exam in minutes	180		
Number of pages including cover page	6	Number of parts / questions	4
If there are multiple versions: VERSION: [ ]			

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I will take this examination to the best of my abilities, without seeking or accepting the help of any source not explicitly allowed by the conditions of the examination.

I have neither given nor received, nor have I tolerated others’ use of unauthorized aid.

Not complying with the statement invalidates the examination for summative use, that is, a grade will not be assigned to your completed examination.

Details (please tick what is applicable by clicking on the selection box)			
Printing (question sheets):	☐ Duplex		☒ Simplex
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Every student is expected to know the examination regulations.

Please write your name and student number on each sheet. Good luck!

Final Exam: Life Insurance (35V6A3-B-6)

Date: 11-06-2024.

Duration: 3 hours.

This exam has 4 pages (not including cover pages).

Give complete answers with sufficient explanations

Question 1 (20=4*5pts): The annual pension benefits for retired civil servants are **50%** of their average salary during their working life (from 25 years to retirement age). According to the new government, the current reserves of the civil service pension fund are far too high. Therefore, the new government wants to lower the current retirement age of 70 by 5 years to the age of 65. Given the huge reserves A_0, the government claims that the current premium rate (that has already been in place for decades) can remain at the same level. The current premium rate equals $\pi\%$ for all civil servants.

Suppose, for simplicity, that the annual salary of civil servants does not change over time (although they might be different among individuals). Moreover, assume a flat term structure of 5%.

(a) After completing a study, someone of the generation $(25, 2024)$ with remaining lifetime $T_{25,2024}$ starts working in one of the government departments. The annual salary of this starter equals $S_1 = S_{\text{starter}}$.

(i) Although the premium rate itself is proposed to remain constant, the proposed change in the retirement age will increase the *net* premium payments of new contracts. Give (in words) at least two reasons why the net premium rate increases.

Solution: Several reasons are possible. Among others, one can explain this by, e.g.,

1) Lowering the retirement age reduces the expected number of premiums paid. The net premiums equal the present value of the expected payoffs (equivalence principle). Thus, net premium increases. ①

2) Lowering the retirement age increases the number of expected pension benefit payments. Thus, the present value of the expected payoff is higher, leading to a higher net premium. ①

(ii) Determine, for the new retirement age 65, an expression for the net premium percentage, say $\pi_{\text{net}}\%$, for someone of the generation $(25, 2024)$ with remaining lifetime $T_{25,2024}$ and annual salary $S_1 = S_{\text{starter}}$. Explain also notation not yet introduced in this question.

Solution: Let $Y(k)$ denote the payoff process, x_{start} the retirement age, and $P_t(t+k)$

the discount factor for the period $[t, t+k]$, then

$$\begin{aligned}\mathbb{E}[Y(k)] &= \frac{1}{2}S_1 \sum_{k=x_{start}-x}^{\infty} \mathbb{P}(T_{x,t} \geq k)P_t(t+k) \\ &= \frac{1}{2}S_1 \sum_{k=40}^{\infty} \frac{{}_kp_{25,2024}}{1.05^k}\end{aligned}$$

where ${}_kp_{25,2024}$ is the k-year survival probability for an individual of age 25 in 2024. ①
By the equivalence principle, $\pi_{net} = \mathbb{E}[Y(k)]$ ①
Consequently, the relative net premium is given by

$$\pi_{net}\% = \frac{1}{2}\frac{\sum_{k=40}^{\infty}\frac{{}_kp_{25,2024}}{1.05^k}}{\sum_{k=0}^{39}\frac{{}_kp_{25,2024}}{1.05^k}} \quad ①$$

(b) The new retirement age will be granted to everyone, not only to (specific) starters on the labor market. Consider a civil servant of the generation $(40, 2024)$ with remaining lifetime $T_{40,2024}$, annual salary S_2, and being in service for 15 years. The new retirement age implies a different payment scheme over time. This can be viewed as a separate, additional contract. Calculate the net premium percentage, say $\pi_{\text{add}}\%$, of this additional contract.

Solution:

$$\begin{aligned}\mathbb{E}[Y(k)] &= \mathbb{E}[Y_{old\,contract}(k)] + \mathbb{E}[Y_{add.}(k)] \quad ① \\ &= \frac{1}{2}S_2 \sum_{k=x_{start,old}-x}^{\infty} \mathbb{P}(T_{x,t} \geq k)P_t(t+k) + \frac{1}{2}S_2 \sum_{k=x_{start,add}-x}^{x_{start,old}-x-1} \mathbb{P}(T_{x,t} \geq k)P_t(t+k) \\ &= \frac{1}{2}S_2 \sum_{k=30}^{\infty} \frac{{}_kp_{40,2024}}{1.05^k} + \frac{1}{2}S_2 \sum_{k=25}^{29} \frac{{}_kp_{40,2024}}{1.05^k} \quad ②\end{aligned}$$

By the equivalence principle, $\pi_{add} = \frac{1}{2}S_2 \sum_{k=25}^{29} \frac{{}_kp_{40,2024}}{1.05^k}$. ①
Consequently, the relative net premium of the additional contract is given by

$$\pi_{add}\% = \frac{1}{2}\frac{\sum_{k=25}^{29}\frac{{}_kp_{40,2024}}{1.05^k}}{\sum_{k=0}^{24}\frac{{}_kp_{40,2024}}{1.05^k}} \quad ①$$

(c) Consider the following four groups of civil servants: (i) starters with age 25, (ii) not yet retired civil servants with age in between 25 and 65, (iii) civil servants with age between 65 and 70, and (iv) retired civil servants with age 70 or older. Discuss for each group the effects by moving to the described new pension plan? Which cohort benefits most? Take also into account the paid premiums. Give several arguments, no formulas needed.

Solution: (i) For starters, the net premium rate increases as discussed in (a). They benefit from the high existing pension fund A_0 though, and they pay five years less premiums. ①

(ii) For civil servants with age 25 to 65, premium rates will remain at the same level, but the older they are, the more they have paid to the reserve. ①

(iii) Civil servants between 65 and 70 years will immediately retire. Instead of paying the premiums for 1 to five years, they will immediately receive pension benefits. ①

(iv) If older than 70, the pension reform does not affect the former civil servants anymore as they have already retired. ①

Consequently, civil servants of the cohort (65,2024) profit the most from this pension reform (five years more pension benefits while not facing the disadvantage of higher net premiums). ①

(d) Consider the pool of current civil servants including retired civil servants. For each individual $i \in \{1, \dots, N\}$ in this pool of N persons, let x_i denote the current age, S_i the (constant) annual salary during service, and T_i the (random) remaining lifetime. Suppose the pension reform with new retirement age 65 starts today.

(i) Give, for each $i = 1, \dots, N$, the corresponding expression of the following random variables: $\mathcal{Y}_i$ the random current value of all possible future pension payments (including the one of today), and $\mathcal{P}_i$ the random current value of all possible future premium payments (including the one of today).

Solution:

$$\begin{aligned}\mathcal{Y}_i &= \sum_{k=0}^{\infty} Y_i(k) P_t(t+k) \\ &= \sum_{k=0}^{\infty} 1_{\{k \geq 65 - x_i\}} 1_{\{T_i \geq k\}} \pi S_i v^k \sum_{k=65-x_i}^{\infty} 1_{\{T_i \geq k\}} \pi S_i v^k \qquad ① \\ \mathcal{P}_i &= \sum_{k=0}^{\infty} \pi(k) P_t(t+k) \\ &= \sum_{k=0}^{64-x_i} \pi_{eff,i} S_i v^k 1_{\{T_i \geq k\}} \qquad ①\end{aligned}$$

where $\pi_{eff,i}$ denotes the % of the salary they effectively pay.

(ii) The probability that the civil service pension fund cannot meet its obligations should be at most 0.1%. Explain in detail how you would check whether the current reserves A_0 are indeed high enough to implement the proposed pension reform. In your discussion, consider that exact calculations are not possible because $N = 997,953$ is fixed but large.

Solution: Simulation approach. Define the asset and liability dynamics

$$A_{k+1} = (1+r_k)\big(A_k - \sum_{i=1}^{N} Y_i(k) + \sum_{i=1}^{N} \pi(k)\big) \qquad ①$$

$$L_k = \sum_{i=1}^{N} \sum_{k=0}^{\infty} \mathbb{E}[Y_i(k)] \qquad (0.5)$$

Simulate the remaining lifetimes of the insureds say 100,000 times along with the returns of the invested assets. (0.5)

Determine the simulated probability that the civil service pension fund cannot meet its obligations by comparing A_k with L_k. If $A_k > L_k$, it can meet its obligations in period k. ①

Question 2 (15=5+10pts):

(a) Consider a swap contract exchanging floating rates with a fixed swap rate during a period of ten years. Explain why the fixed swap rate depends on the payment frequency, say every month, every quarter, every year, or ....

Solution: The fixed swap rate is given by

$$r = \frac{1 - P_t(T_n)}{\sum_{i=1}^{n} P_t(T_i)\Delta T} \qquad ①$$

This swap rate is defined as the rate that makes the swap contract fair at initiation, i.e., it equalizes the present value of the fixed leg and the floating leg. ①

Fixed payments: $r \cdot \Delta T \cdot N$. ①

Floating payments: $\big(\frac{1}{P_{T_i-1}(T_i)} - 1\big)N$. ①

Both legs depend on the payment frequency, and hence the swap rate also depends on that payment frequency. ①

(b) In the Vasicek model the term structure of interest rates is explained by a single factor $\{X_t\}$ with dynamics:

$$X_{t+1} = \mu + \theta X_t + \sigma\varepsilon_{t+1}, \text{ with } \varepsilon_{t+1} \sim N(0,1) \ i.i.d., \ \theta \in [0,1), \ \sigma > 0.$$

For each $k \in \mathbf{N}$, it can be shown that the current yield of a zero coupon bond with maturity date $t+k$ is given by $R_t(t+k) = (1 - \frac{B_k}{k})(\frac{\mu^{\mathbb{Q}}}{1-\theta} - \frac{1}{2}\frac{\sigma^2}{(1-\theta)^2}) + \frac{B_k}{k}X_t + \frac{1}{2}\sigma^2\frac{1}{1+\theta}\frac{1}{1-\theta}\frac{B_k(B_k-1)}{k}$, where $B_0 = 0$ and $B_k = 1 + \theta B_{k-1} = \frac{1-\theta^k}{1-\theta}$.

(i) Interpret the factor X_t.

Solution: X_t is the short-term spot rate holding for a period of length ΔT, i.e., it is revealed at period t and holds for an investment from t to $t+1$. ②

(ii) Show that the limit of $R_t(t+k)$ when $k \to \infty$ does not depend on X_t. Also determine this limit.

Solution: Notice that $\lim_{k\to\infty} B_k = \frac{1}{1-\theta}$ and $\lim_{k\to\infty} \frac{B_k}{k} = 0$. ①

Consequently,

$$\begin{aligned}\lim_{k\to\infty} R_t(t+k) &= \lim_{k\to\infty} (1-\frac{B_k}{k})(\frac{\mu^{\mathbb{Q}}}{1-\theta} - \frac{1}{2}\frac{\sigma^2}{(1-\theta)^2}) + \frac{B_k}{k}X_t + \frac{1}{2}\sigma^2\frac{1}{1+\theta}\frac{1}{1-\theta}\frac{B_k(B_k-1)}{k} \\ &= \frac{\mu^{\mathbb{Q}}}{1-\theta} - \frac{1}{2}\frac{\sigma^2}{(1-\theta)^2}. \quad ①\end{aligned}$$

(iii) When $\theta = 0$, it is possible that all yields are equal at time t. Which constraint is needed to obtain equal yields?

Solution: If $\theta = 0$, bond yields are given by

$$R_t(t+k) = \left(1 - \frac{1}{k}\right)(\mu^{\mathbb{Q}} - \frac{1}{2}\sigma^2) + \frac{X_t}{k} \quad ①$$

This expression if independent of k if and only if $X_t = \mu^{\mathbb{Q}} - \frac{1}{2}\sigma^2$ leading to a flat term structure of interest rates at time t. ①

(iv) Do you think that $\theta = 0$ results in a reasonable model for X_t? Why (not)?

Solution: For $\theta = 0$, the short-term spot rate follows a (shifted) white noise process and does not have a mean-reverting feature. ①

However, there is a lot of evidence, that interest rates tend to have a mean reverting property. Thus $\theta = 0$ is not realistic. ①

(v) Explain what is meant by a flat term structure. Show that a flat term structure is also not possible if $0 < \theta < 1$.

Solution: A term structure is flat if bond yields are independent of their time-to-maturity, i.e., if $R_t(t+k)$ is independent of k. ①

It is obvious that $R_t(t+1) \neq \frac{\mu^{\mathbb{Q}}}{1-\theta} - \frac{1}{2}\frac{\sigma^2}{(1-\theta)^2}$ if $\theta \neq 0$, i.e., short term interest rates differ from their long-term limit. ①

***** Start Questions 3 and 4 on a new answer sheet *****

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Question 3 (22=4+4+6+4+4 pts):
The *raw central death rate* for a cohort (x, t) is formally defined as

$$m_{x,t} = \frac{D_{x,t}}{E_{x,t}}$$

where $D_{x,t}$ denotes the *number of death* of an age group x between year t and $t+1$, and $E_{x,t}$ is the *exposure.*

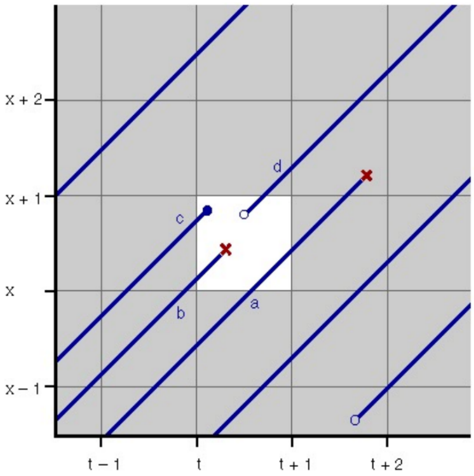
Hint: To simplify the notation, we suppress the superscript (g) referring to a group or gender.

(a) State a definition of the exposure and illustrate your explanations with a suitable Lexis diagram that contains at least four lives.

Solution: Let $n_{x,t}^{(g)}$ lives contribute to the observations in the white square (Lexis diagram). Assume life $i \in N_{x,t}^{(g)} = \{1, \dots, n_{x,t}^{(g)}\}$ is observed between the ages of $x + t_i$ and $x + s_i$ (with $0 \le t_i < s_i \le 1$). ①

The *exposure* $E_{x,t}^{(g)}$ is defined as

$$E_{x,t}^{(g)} = \sum_{i=1}^{n_{x,t}^{(g)}} (s_i - t_i). \qquad ①$$



②

(b) Assuming that the remaining lifetimes $T_{x,t}$ of all individuals are stochastically independent, the likelihood of $n_{x,t}$ observations in cohort (x, t) can be expressed in terms of the *hazard rate of death* $\mu_{x,t}$:

$$L_{x,t} = \prod_{i \in I_{x,t}} \mu_{x,t} \prod_{i \in N_{x,t}} \exp\big(- (s_i - t_i)\mu_{x,t}\big).$$

Prove that the raw central death rate is the maximum likelihood estimator for the hazard rate of death, i.e., $\widehat{\mu}_{x,t}^{ML} = m_{x,t}$.

Solution: the likelihood can be rewritten:

$$\begin{aligned} L_{x,t}^{(g)} &= \prod_{i \in I_{x,t}^{(g)}} \mu_{x,t}^{(g)} \prod_{i \in N_{x,t}^{(g)}} \exp\Big(-(s_i - t_i)\mu_{x,t}^{(g)}\Big) \\ &= \big[\mu_{x,t}^{(g)}\big]^{D_{x,t}^{(g)}} \exp\Big(- \sum_{i \in N_{x,t}^{(g)}} (s_i - t_i)\mu_{x,t}^{(g)}\Big) \qquad ① \\ &= \big[\mu_{x,t}^{(g)}\big]^{D_{x,t}^{(g)}} \exp\Big(- E_{x,t}^{(g)}\mu_{x,t}^{(g)}\Big) \\ &= \exp\Big(- E_{x,t}^{(g)}\mu_{x,t}^{(g)} + D_{x,t}^{(g)} \log(\mu_{x,t}^{(g)})\Big) \qquad ① \end{aligned}$$

Determining the log-likelihood function

$$\ell_{x,t}^{(g)} = -E_{x,t}^{(g)}\mu_{x,t}^{(g)} + D_{x,t}^{(g)} \log(\mu_{x,t}^{(g)}) \qquad ①$$

Maximizing w.r.t. $\mu_{x,t}^{(g)}$ yields

$$\mu_{x,t}^{ML} = m_{x,t}. \qquad ①$$

(c) Suppose now that a pension fund manager wants to model the hazard rate of death of a population assuming that the remaining lifetime $T_{x,t}$ of a cohort is Weibull-distributed with age-dependent parameters $\lambda = \lambda_x$, $\theta = \theta_x$ and probability density function

$$f_{x,t}(z) = \lambda\theta(\lambda z)^{\theta-1}\mathrm{e}^{-(\lambda z)^\theta}.$$

(i) Determine the one-year survival probability and the force of mortality in this model.
Solution:

- Let $T_{x,t}^{(g)}$ denote the (random) remaining lifetime of somebody of age x, group g, at time t with probability distribution function

$$F_{x,t}^{(g)}(\tau) = \mathbb{P}(T_{x,t}^{(g)} \le \tau) = 1 - \mathrm{e}^{-(\lambda z)^\theta}. \qquad ②$$

- 1y survival probability

$$p_{x,t}^{(g)} = \mathrm{e}^{-(\lambda z)^\theta}. \qquad ①$$

- The force of mortality (or hazard rate of death) is

$$\mu_{x+\tau,t+\tau}^{(g)} = \frac{f_{x,t}^{(g)}(\tau)}{1 - F_{x,t}^{(g)}(\tau)} = \lambda\theta(\lambda z)^{\theta-1}. \qquad ①$$

(ii) Explain why this model ignores macro-longevity risk.

Solution: Survival probabilities are deterministic, hence there is no uncertainty about their future evolution. ①

(iii) Find a way to adjust this model such that it accounts for macro longevity risk.

Solution: Many ways are imaginable, e.g., modeling λ as a stochastic processes of the Cox-Ross-Rubinstein type. ①

(d) The Lee-Carter model is the benchmark model for macro longevity risk. Explain its components and how macro longevity risk is modeled. What needs to be done to identify the parameters in this model?

Solution:

- Central death rate evolves stochastically according to the dynamics

$$\ln(m^{(g)}_{x,t}) = \alpha^{(g)}_x + \beta^{(g)}_x \kappa^{(g)}_t + \varepsilon^{(g)}_{x,t},$$ ①

where $\varepsilon^{(g)}_{x,t} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\varepsilon^{(g)}})$. $\alpha^{(g)}_x$ and $\beta^{(g)}_x$ are age-specific parameters. ①

- Dynamics of the latent factor, typically modeled as a random walk with drift:

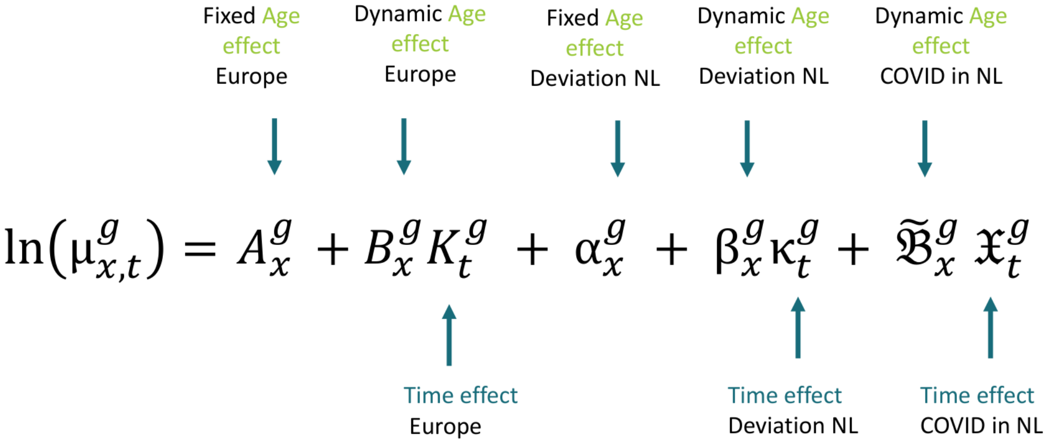
$$\kappa^{(g)}_t = c^{(g)} + \kappa^{(g)}_{t-1} + \delta^{(g)}_t,$$ (0.5)

where $\delta^{(g)}_t \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\delta^{(g)}})$. (0.5)

- Model is overparametrized and one need to choose an appropriate normalization that reduces the degrees of freedom. ①

(e) As in the AG2022 model, one could directly model the hazard rate of death. Briefly explain the three layers of the AG2022 model and elaborate on the estimation technique.

Solution: The AG2022 model models three layers (Europe, NL deviation, and COVID in NL) to describe the log hazard rate of death:



③

This model can be estimated with maximum likelihood as

$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}\Big(E_{x,t}^{(g)} \mu_{x,t}^{(g),pre-covid}\Big). \qquad ①$$

Question 4 (13=1+4+5+3 pts):
A pension fund manager would like to quantify the risk in the funding ratio until the year 2029. The fund manager asks you to propose an appropriate model-based (simulation) approach to quantify this risk, taking into account interest rate risk, micro and macro longevity risk. The fund manager wants you to use the AG2022 model. You may ignore stock market risk and model risk.

(a) Explain what the funding ratio is and why it is an important notion.

Solution:

$$FR = A/L \qquad \textcircled{0.5}$$

It indicates whether the fund is able to pay all its liabilities. (0.5)

(b) Write down the laws of motion of the pension fund's assets and its liabilities and explain how they are affected by macro longevity risk.

Solution:

$$A_{t+1} = A_t(1+r_t) - \sum_{x\in\mathcal{X}} N_{x+1,t+1}. \qquad \textcircled{2}$$

$$L_{t+1}^{BE(t+1)} = (1+r_t)L_t^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x+1,t+1}. \qquad \textcircled{2}$$

(c) Elaborate on how to simulate the number of survivors $N_{x+1,t+1}$ given $N_{x,t}$ for $t = 0$ (2024). What is different in later years $t = 1, \ldots, 4$ (2025,. . . 2029) and how would you account for this difference? Relate your answer to the AG2022 model.

Solution: At $t = 0$, the best-estimate death probability is not uncertain, and thus $N_{x+1,t+1}$ is binomial distributed. ①

Simulate a large number (e.g., 100,000) of random variables drawn from a bin distribution with parameters $p_{x,t}^{BE}$ and $N_{x,t}$. ①

In the subsequent years, the survival probabilities need to be simulated from the AG2022 model and are then path-dependent as they explicitely depend on the number of survivors in the previous period, which is a random variable itself. Thus, the future survivors are not binomially distributed. ②

The number of deaths can be simulated by drawing from a Poisson distribution conditional on the observed exposure. ①

(d) To simplify the simulation, you propose the following recursive relation for the annuity factor.

$$a^{BE(t+1)}_{x+1,t+1} = (1 + r_t)\frac{a^{BE(t)}_{x,t}}{p^{BE(t)}_{x,t}} - 1.$$

Explain briefly why it is useful to make use of this representation in your simulations when we face interest rate risk, micro and macro longevity risk.

Solution: Using this recursive relation, we can find an easy-to-simulate representation of liabilities

$$\begin{aligned} L^{BE(t+1)}_{t+1} &= \sum_{x\in\mathcal{X}} N_{x+1,t+1} a^{BE(t+1)}_{x+1,t+1} \qquad \text{①} \\ &= \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t} a^{BE(t+1)}_{x+1,t+1} \qquad \text{①} \end{aligned}$$

where

$$a^{BE(t+1)}_{x+1,t+1} = \sum_{\tau=1}^{\infty} {}_{\tau}p^{BE(t+1)}_{x+1,t+1} \prod_{j=0}^{\tau-1} \frac{1}{1 + r_{t+1+j}}. \qquad \text{①}$$