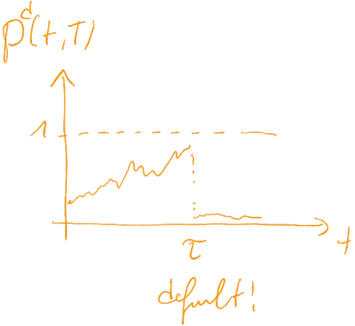
Part VII

A Brief Introduction to Credit Risk


$p^d(t,T)$
1
τ
t
default!

Motivation

- So far, we have considered discount factors and term structures related to default-free bonds.
- In reality there is always credit risk, i.e., the risk of default from an issuer of a bond (the borrower) failing to make the payments

Definition: Credit Risk

Credit risk is the risk that the holder of a financial asset experiences a loss because of

- a debtor's non-payment of a loan or other line of credit (either the principal or interest (coupon) or both)
- a default by the counterparty in a derivatives transaction.

- Credit risk differs from market risk since
  - default is a 0-1-event
  - default risk is harder to measure
  - default risk cannot be hedged away by a market index

How to Quantify Credit Risk?

τ : default time

e.g. $N = 100$

$L_\tau = 40\%$

$R_\tau = 60\%$

- There are two dimensions of credit risk:
  1. How likely is a default?
  2. How big is the loss if a default occurs?
- These dimensions are captured by the
  1. default probability (PD),
  2. loss given default (LGD), L_τ.
- Recovery rate $R_\tau = 1 - L_\tau$
- Can these quantities be identified from historical data? For instance, BASF has never defaulted. Does this mean that its default probability is zero?
- **Idea:** Back out credit risk from the prices of credit derivatives and corporate bonds.

$$P_t^d(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\left(\mathbb{1}_{\{\tau > T\}} + R_\tau \mathbb{1}_{\{\tau \leq T\}}\right)\right]$$

Intensity Model

- We are now going to introduce discount factors corresponding to defaultable zero coupon bonds.
- Let the defaultable zero coupon bond's maturity be T and its face value be 1. Denote its value at time $t \leq T$ by $P_t^d(T)$.
- Modeling credit risk is usually done by introducing a random (first) default time $\tau \in \mathbb{R}^+$.
  - In case of no default ($\tau > T$), the bond pays off 1 at time T.
  - In case of default ($\tau \leq T$), the bond pays off $R_\tau = 1 - L_\tau$ at time T.

  Here $L_\tau \in (0, 1]$ is the loss rate.
- The default time τ is modeled as the first jump of a counting process (typically a Poisson or a Cox process) $N_t \in \mathbb{N}$, i.e., $N_0 = 0$

$$\tau = \min\{t \mid N_t = 1\}$$

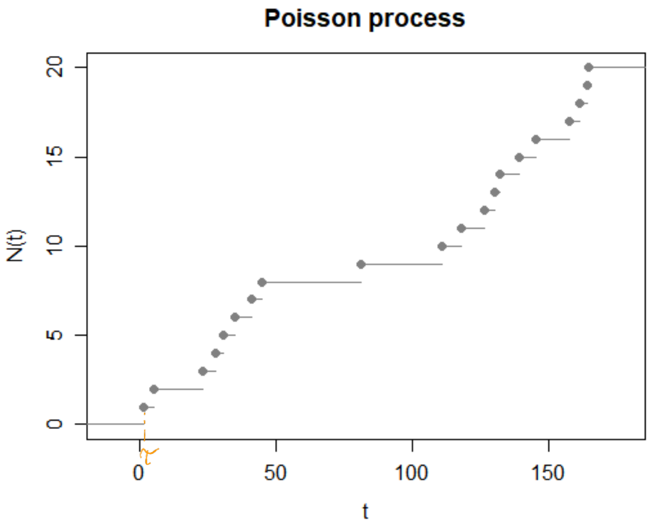
Poisson and Cox Processes

- A *Poisson process* N is an increasing process taking values in $\mathbb{N}$ (a so-called counting process) with
  1. $N_0 = 0$ $\quad [t, s] \qquad [u, v]$
  2. independent increments $\quad N_s - N_t$ is stoch indepndt of $N_v - N_u$
  3. the number of events (or points) in any interval of length t is a Poisson random variable with mean λt. $\quad X \sim \mathcal{P}(\lambda) \Rightarrow \mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$
- The parameter λ is called the jump intensity (or *default intensity*, or *hazard rate*) and models the instantaneous default probability, i.e.,

$$\lambda = \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_{t+\Delta t} > N_t)}{\Delta t}$$

- If the parameter λ is itself a non-negative stochastic process, we call N a Cox process. A typical choice is that λ is of the CIR type, i.e.,

$$\mathrm{d}\lambda_t = a(b - \lambda_t)\mathrm{d}t + \sigma \sqrt{\lambda_t}\,\mathrm{d}W_t$$


Poisson process
N(t)
0
5
10
15
20
0
50
100
150
t
τ

Interpretation of Default Intensity

- Consider a Poisson process $N^{\mathbb{Q}}$ with intensity $\lambda^{\mathbb{Q}}$ under $\mathbb{Q}$. Default happens if the first jump of N happens before maturity.
- Probability of default under $\mathbb{Q}$

$$\mathbb{Q}(\tau \leq T) = \mathbb{Q}(N_T \geq 1) = 1 - \mathbb{Q}(N_T = 0) =_{(3)} 1 - \mathrm{e}^{-\lambda^{\mathbb{Q}} T}$$

- In particular, the one-year default probability is

Taylor expansion:
$e^{-x} \approx 1 - x$

$$\mathbb{Q}(\tau < 1) = 1 - \mathrm{e}^{-\lambda^{\mathbb{Q}}} \approx \lambda^{\mathbb{Q}}$$

- Consequently, the default intensity is approximately the one-year probability of default.
- In reality, default probabilities are not constant, but depend on macroeconomic indicators and firm-specific variables.

- Standing assumption: Default intensity λ_t, short rate r_t, and recovery rate R_t are stochastically independent.
- Under this assumption, interest rate risk can be disentangled from default risk.

$$
\begin{aligned}
P_0^d(T) &= \mathbb{E}^{\mathbb{Q}}\left[\overbrace{e^{-\int_0^T r_s ds} 1_{\{\tau > T\}}}^{\text{no default}} + \overbrace{e^{-\int_0^T r_s ds} 1_{\{\tau \le T\}} R_\tau}^{\text{default}}\right] \\
&\overset{\text{stoch. indep.}}{=} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\right]\mathbb{Q}(\tau > T) + \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T r_s ds}\right]\mathbb{Q}(\tau \le T)\mathbb{E}^{\mathbb{Q}}[R_\tau] \\
&= P_0(T)\Big(\mathbb{Q}(\tau > T) + \mathbb{Q}(\tau \le T)\mathbb{E}^{\mathbb{Q}}[R_\tau]\Big) \\
&= P_0(T)\Big(1 - \overbrace{E^{\mathbb{Q}}[L_\tau]\mathbb{Q}(\tau \le T)}^{\text{compensation for credit risk}}\Big) \\
&= P_0(T)\Big(1 - E^{\mathbb{Q}}[L_\tau]\big(1 - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \lambda_s^{\mathbb{Q}} ds}\right]\big)\Big)
\end{aligned}
$$

Credit Spread

- The credit spread between both bonds:

$P_0(T) = e^{-R_0(T)T}$

$\Rightarrow R_0(T) = -\frac{1}{T} \log P_0(T)$

$$
\begin{aligned}
S_0^d(T) &= R_0^d(T) - R_0(T) \\
&= -\frac{1}{T} \log P_0^d(T) + \frac{1}{T} \log P_0(T) \\
&= -\frac{1}{T} \log \left(1 - E^{\mathbb{Q}}[L_\tau]\left(1 - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \lambda_s^{\mathbb{Q}} ds}\right]\right)\right) \\
&\approx \frac{1}{T} E^{\mathbb{Q}}[L_\tau]\left(1 - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \lambda_s^{\mathbb{Q}} ds}\right]\right)
\end{aligned}
$$

$\log(1 \pm x) \approx \pm x$

- If the default intensity λ is constant:

$e^{-x} \approx 1 - x$

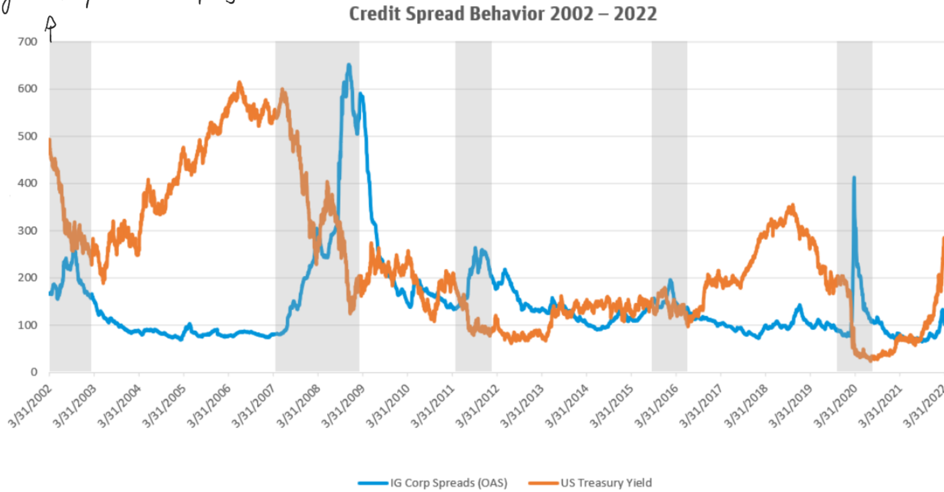
$$
\begin{aligned}
S^d(T) &\approx \frac{1}{T}\mathbb{E}^{\mathbb{Q}}[L_\tau](1 - \mathrm{e}^{-\lambda^{\mathbb{Q}} T}) \\
&\approx \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}[L_\tau]
\end{aligned}
$$

- **Rule of thumb:** Yield spread between corporate bond and Treasury bond approximately equals the expected one-year loss due to default risk under the risk-neutral measure.

Credit Spread

100 bps = 1%

yield / spread [bps]



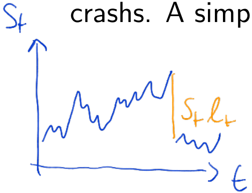
10y BBB-rated corp bonds.   10y Treasury bonds

Some Remarks

- A thorough quantitative analysis of credit risk requires Itô calculus with jump processes.
- Term structure equations become more complicated as they involve jump terms.
- If both the short rate process and the intensity process are affine, then the corporate bond prices before default are affine as well, i.e.,

$$P_t^d(T)1_{\{t<\tau\}} = \mathrm{e}^{A^d(t,T)+B^d(t,T)r_t+C^d(t,T)\lambda_t}$$

- Jump processes are also commonly used to model stock market crashs. A simple example is the Merton Jump-Diffusion model



$$\mathrm{d}S_t = S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t + \boxed{S_t\ell_t}\mathrm{d}N_t.$$

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Not relevant for examination

Idea: Merton's Firm Value Model

Merton's Firm Value Model

- Firm has debt – modeled by a zero bond with
  - notional F
  - maturity at time T
  - default only at time T possible
- At T: Redemption depends on the firm value V_T

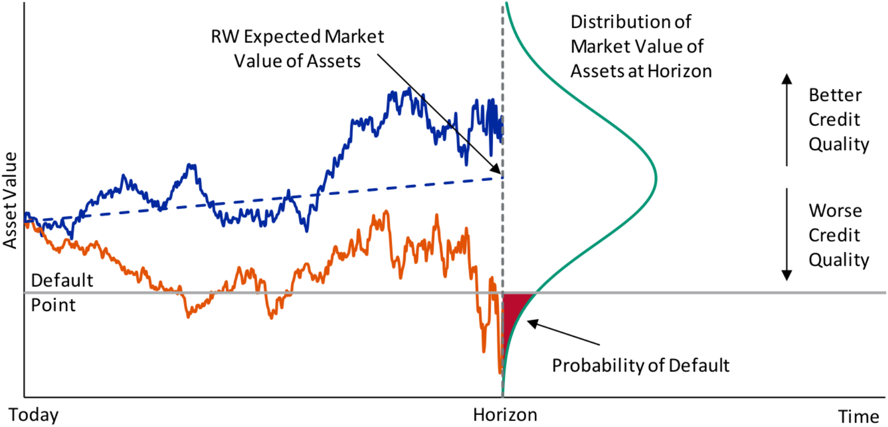
$$D_T = \min\{V_T, F\}$$

If $V_T < F$: default.
$\Longrightarrow$ Loss given default: $L = F - V_T$

- Shareholders get the residuum

$$\begin{aligned} E_T &= V_T - D_T \\ &= V_T - \min\{V_T, F\} \\ &= \max\{V_T - F, 0\} \end{aligned}$$

$\Longrightarrow$ Equity is a call option on the firm value with maturity at time T and strike price F.



Source: Moody's Research Analytics

Merton's Firm Value Model

- Model the firm value like the stock price in the Black-Scholes model (V is log-normally distributed)
- Equity is a call option on the firm value
  $\Longrightarrow$ Black-Scholes formula delivers:

$$
\begin{aligned}
E_0 &= V_0 \Phi(d_1) - F e^{-rT} \Phi(d_2) \\
D_0 &= V_0 - E_0 = F e^{-R^d(T)T} \\
d_1 &= \frac{\ln(V_0/F) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

- Credit spread:

$$
S_0(T) = \frac{1}{T} \log\left(\frac{F}{D_0}\right) - r
$$

Merton's Firm Value Model

- Weaknesses
  - Same weaknesses as the Black-Scholes model (e.g., constant volatility, interest rates)
  - V is typically not traded (but E).$\Longrightarrow$ How do we know σ?

$$\sigma \frac{\Phi\big(d_1(\sigma)\big)}{E(\sigma)} = \frac{\sigma_E}{V}$$

  - Very simplistic debt policy. Firms do not emit just one zero bond. In reality, they emit several coupon bonds, mortgages, and other forms of credit contracts with different maturities.
- However, economic implications are quite plausible.
- Firm value model acts as a building block for many practically-relevant models (e.g., Moody's KMV Model, J.P. Morgans' Credit Metrics, ...)
- Popular alternative model in credit risk management: Credit Risk+