Part III

Contingent Claim Pricing

European / American Options

- This chapter studies examples for contingent claim pricing in several tangible specifications of the GSSM.

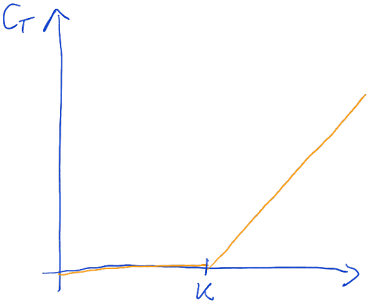
Option

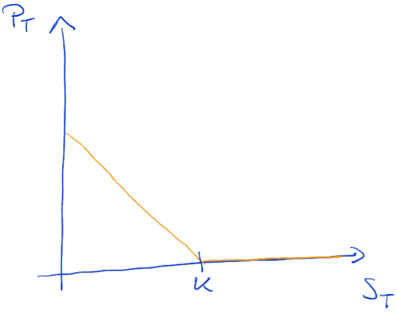
1. A European option is a contract between two counterparties, whereby the buyer (= holder) has the right to buy (= Call option) or to sell (= Put option) the underlying from/to the seller (= stillholder) for a predetermined strike price K *at* its maturity T.
2. An American option has the feature that the option can be exercised *before* maturity, i.e., in $[0, T]$.

- Option profile at maturity T on a stock with price process S:

$$\begin{aligned} C_T &= (S_T - K)^+ = \max\{S_T - K, 0\} \\ P_T &= (K - S_T)^+ = \max\{K - S_T, 0\} \end{aligned}$$

Option Profiles





$$C_T = (S_T - K)^+$$

$$P_T = (K - S_T)^+$$

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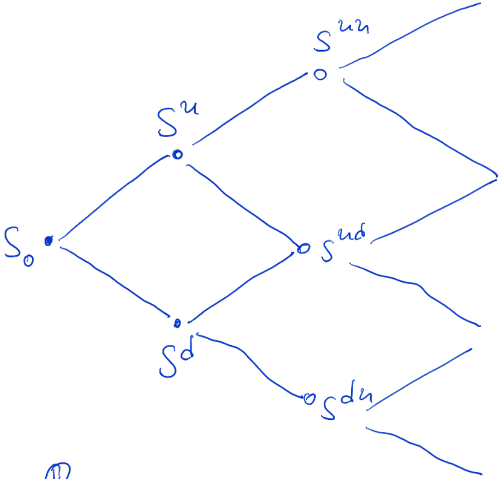
Not relevant for examination

- The Black-Scholes formula is probably the most famous formula in quantitative finance and the starting point of modern financial mathematics.
- Black and Scholes (1973) derive the formula by transforming the BSPDE to the heat equation, which has a well-known solution

$$r\,\pi_C = \frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} Sr + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2} S^2 \sigma_S^2$$

  s.t. $\pi_C(T, S_T) = \max(S_T - K, 0)$.
- Besides solving the BSPDE, the problem can be tackled by several approaches, e.g.,
  - Pricing under the EMM $\mathbb{Q}$
  - Pricing under $\mathbb{P}$ using the SDF / numéraire portfolio
  - Taking the limit of a sequence of binomial models
  - Splitting the payoff into two parts and tackle them under two different measures
  - ...



Binomial tree models converge to the BS model

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{e^{rT}}\right]$$

The Fastest Way to the Black-Scholes Formula

- The European call option has payoff function

$$C_T = \max(S_T - K, 0) = 1_{\{S_T \geq K\}}(S_T - K).$$

- The price of the European put option with payoff $P_T = \max(K - S_T, 0)$ can be obtained from the put-call parity

$$P_t = C_t - S_t + K\mathrm{e}^{-r(T-t)}.$$

- We can decompose the call option into two options;
  1. a long position in the *stock-or-nothing option* which has payoff $1_{\{S_T \geq K\}} S_T$
  2. a short position in the *cash-or-nothing option* which has payoff $1_{\{S_T \geq K\}} K$.
- The price of the call option is determined if we know the prices of the stock-or-nothing option and the cash-or-nothing option.

- Cash-or-nothing option, $C_T^m = \underline{1_{\{S_T \geq K\}} K}$ will be priced under $\mathbb{Q}$:

NDPF

$$\boxed{\frac{C_0^m}{M_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T^m}{M_T}\right]} \overset{M_0=1}{=} \frac{K}{M_T}\,\mathbb{E}^{\mathbb{Q}}\left[1_{\{S_T \geq K\}}\right] = \boxed{\frac{K}{M_T}}\,\boxed{\mathbb{Q}(S_T \geq K).}$$

- Under $\mathbb{Q}$, the evolution of the stock price is given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

- Therefore:

$$S_T = S_0 \exp\left(\left(r - \tfrac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}Z\right), \qquad Z \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\Longrightarrow \mathbb{Q}(S_T \geq K) = \Phi(d_2), \quad d_2 = \frac{\log(S_0/K) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Stock-or-nothing option under $\mathbb{Q}_S$

- Stock-or-nothing option, $C_T^s = 1_{\{S_T \geq K\}} S_T$:

$$\text{NDPF: } \boxed{\frac{C_0^s}{S_0} = E^{\mathbb{Q}_S}\left[\frac{C_T^s}{S_T}\right]} = E^{\mathbb{Q}_S}\left[1_{\{S_T \geq K\}}\right] = \mathbb{Q}_S(S_T \geq K).$$

- Under $\mathbb{Q}_S$, the evolution of the stock price is given by

$$\mathrm{d}S_t = (r + \sigma^2) S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t^{\mathbb{Q}_S}$$

where $W^{\mathbb{Q}_S}$ is a Brownian motion under $\mathbb{Q}_S$.

- Therefore:

$$S_T = S_0 \exp\left((r + \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T} Z\right), \qquad Z \overset{\mathbb{Q}_S}{\sim} N(0,1)$$

$$\Longrightarrow \mathbb{Q}_S(S_T \geq K) = \Phi(d_1), \quad d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}}.$$

The Black-Scholes Formula

$$C_0 = S_0 \, \mathbb{Q}^S(S_T \geq K) - DF \cdot K \cdot \tilde{\mathbb{Q}}(S_T \geq K)$$

- Putting everything together:

$$C_0 = C_0^s - C_0^m = S_0\Phi(d_1) - \boxed{e^{-rT}K}\Phi(d_2).$$

- The price of the call option is equal to the current value of the stock times the probability under $\mathbb{Q}_S$ that the option will end in the money, minus the current value of the strike times the probability under $\mathbb{Q}$ that the option will end in the money.

Problem: Derive the $\mathbb{Q}_S$-Stock Dynamics

- $\frac{S}{S}$, $\frac{M}{S}$ need to be martingales under $\mathbb{Q}_S$

- $\mathbb{P}$: $dS = S[\mu dt + \sigma dW]$
  $\mathbb{Q}$: $dS = S[r dt + \sigma dW^{\mathbb{Q}}]$
  $\mathbb{Q}^S$: $dS = S[\mu^{\mathbb{Q}_S} dt + \sigma dW^{\mathbb{Q}_S}]$

- $d\left(\frac{M}{S}\right) = M\, d\left(\frac{1}{S}\right) + \frac{1}{S} dM + \underbrace{d\left[M, \frac{1}{S}\right]}_{=0}$

$$d\left(\frac{1}{S}\right) \overset{\text{Itô}}{=} -\frac{1}{S^2} dS + \frac{1}{2}\cdot\frac{2}{S^3} d[S]$$
$$= -\frac{1}{S^{\cancel{2}}}\cdot \cancel{S}\left[\mu^{\mathbb{Q}_S} dt + \sigma dW^{\mathbb{Q}_S}\right] + \frac{1}{S^{\cancel{3}}} \cancel{S^2} \sigma^2 dt$$
$$= \frac{1}{S}\left[\left(-\mu^{\mathbb{Q}_S} + \sigma^2\right) dt - \sigma dW^{\mathbb{Q}_S}\right]$$

Problem: Derive the $\mathbb{Q}_S$-Stock Dynamics

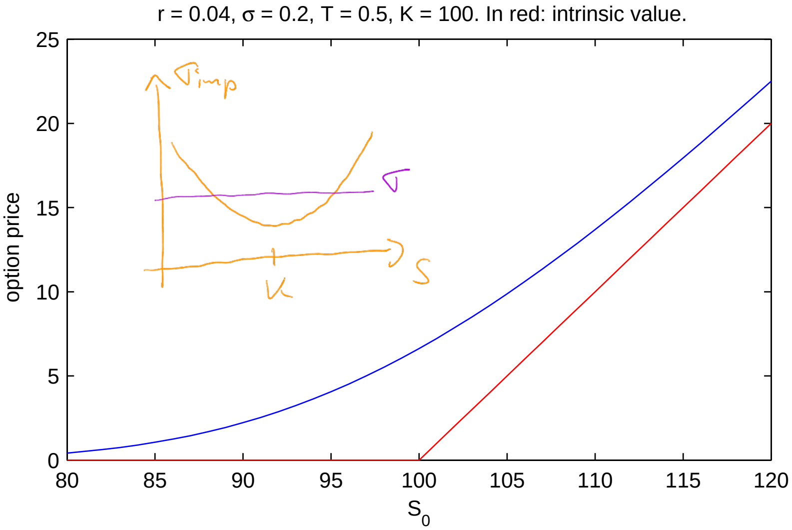
$$d\left(\frac{M}{S}\right) = \frac{M}{S}\left[\left(-\mu^{\mathbb{Q}_S}+\sigma^2\right)dt + \sigma dW^{\mathbb{Q}_S} + r\,dt\right]$$

$$\Rightarrow -\mu^{\mathbb{Q}_S} + \sigma^2 + r = 0$$

$$\Rightarrow \boxed{\mu^{\mathbb{Q}_S} = \sigma^2 + r}$$

$$\Rightarrow dS = S\left[\left(r+\sigma^2\right)dt + \sigma dW^{\mathbb{Q}_S}\right]$$

Option Price versus Intrinsic Value

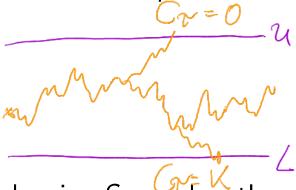


Critique: Black-Scholes Model

- Volatility, interest rate, expected return are assumed to be constant.
  $\longrightarrow$ Volatility Smile
- Returns are assumed to be normally distributed. $\longrightarrow$ Underestimation of extreme events.
- Model builds upon a complete market without frictions (no taxes, transaction costs, short-selling constraints, ...).
- Implied volatility $\neq$ historical volatility
  - These caveats become visible if one investigates what volatilities are necessary to explain option prices by the Black-Scholes formula.
  - Implied volatility is not constant, but depends on K and T.
  - If the option is at-the-money, implied volatility is lowest (volatility smile).
- Some of these points can be tackled by adding non-traded state variables to the model.

A Double-barrier Option

- A *perpetual up-and-out down-and-in digital double barrier option* is a contract that is specified by
  - an underlying S_t (for instance a stock index)
  - a lower barrier L
  - an upper barrier U
  - a fixed payoff amount K.



- The contract pays the amount K when the stock price S_t reaches the lower barrier L, but only if the stock price has not reached the upper barrier first. (i.e., the contract "knocks out" when the stock price S_t reaches U.)
- As long as neither the lower nor the upper barrier has been reached, the contract stays alive.
- Therefore the time of expiry of the contract is random (determined in terms of the process S_t).

PDE Approach

- Assume that the BS model holds for the stock price S_t. The Black-Scholes equation for the pricing function $\pi_C(t, S_t)$ is in general

$$\cancel{\frac{\partial \pi_C}{\partial t}(t, S)} + rS \frac{\partial \pi_C}{\partial S}(t, S) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \pi_C}{\partial S^2}(t, S) = r\pi_C(t, S).$$

- Since π_C does not depend on t, this reduces to the ODE

$$rS \frac{\mathrm{d}\pi_C}{\mathrm{d}S}(S) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 \pi_C}{\mathrm{d}S^2}(S) = r\pi_C(S).$$

- Boundary conditions for the up-and-out down-and-in option:

$$\boxed{\pi_C(U) = 0,} \qquad \boxed{\pi_C(L) = K.}$$

$\pi_C = S$

Solving the ODE

- We have a linear homogeneous second-order ODE, so the general solution is a linear combination of two particular solutions.
- These solutions should be self-financing portfolios whose values depend only on S_t. One solution is $\boxed{S_t}$ itself (obviously!), another is $S_t^{-\gamma}$ with $\gamma = 2r/\sigma^2$. → Substitute into the ODE and check!
- The solution is therefore given by

$$\pi_C(S_t) = c_1 S_t + c_2 S_t^{-\gamma}$$

where the constants c_1 and c_2 should be chosen such that

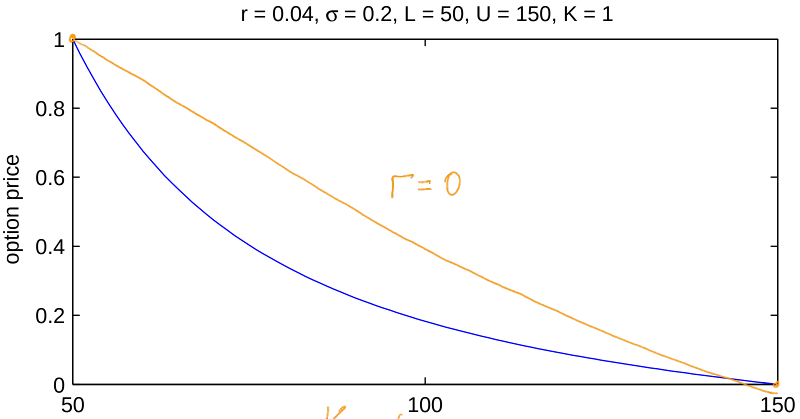
$$\pi_C(U) = c_1 U + c_2 U^{-\gamma} = 0, \quad \pi_C(L) = c_1 L + c_2 L^{-\gamma} = K.$$

- This linear system has a unique solution.

- Putting everything together yields $\pi_C = c_1 S + c_2 S^{-\gamma}$

$\gamma = \frac{2r}{\sigma^2}$

$$\pi_C(t, S_t) \;=\; \frac{L^\gamma K}{U^{\gamma+1} - L^{\gamma+1}}\left(U^{\gamma+1} S_t^{-\gamma} - S_t\right).$$



$r = 0$: $\gamma = 0$: $\pi_C = \frac{K}{U - L}(U - S_t)$

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An Example: The Heston Model

- Modeling Stochastic Volatility by a CIR process

$$
\begin{aligned}
dM_t &= M_t r dt \\
dS_t &= S_t[\mu dt + \sqrt{\nu_t} dW_{1,t}] \\
d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} d(\rho \boxed{W_{1,t}} + \sqrt{1-\rho^2} W_{2,t})
\end{aligned}
$$

r

$W_1^{\mathbb{Q}} + \lambda_1 t$ $\quad$ $W_2^{\mathbb{Q}} + \lambda_2 t$

$\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - \nu_t) \Rightarrow$ guarantees that $\nu_t > 0$

- The model has five input parameters:
  - ν_0, the initial variance.
  - θ , the mean-reversion variance of the stock price
  - κ, the mean-reversion speed of the variance of the stock price
  - ρ the correlation of the two Wiener processes.
  - σ the volatility of the volatility, or 'vol of vol'

r: interest rate

- $n = 3$ state variables, $k = 2$ sources of risk, and $m = 2$ assets:

$$
\mu_X = \begin{bmatrix} \mu S_t \\ r M_t \\ \kappa(\theta - \nu_t) \end{bmatrix}, \quad \sigma_X = \begin{bmatrix} \sqrt{\nu_t} S_t & 0 \\ 0 & 0 \\ \sigma\rho\sqrt{\nu_t} & \sigma\sqrt{\nu_t}\sqrt{1-\rho^2} \end{bmatrix}, \quad \pi_Y = \begin{bmatrix} S_t \\ M_t \end{bmatrix}
$$

Economic Properties

- The model is free of arbitrage: The NA criterion $\mu_Y - \pi_Y r = \sigma_Y \lambda$ yields

$$\begin{array}{ccccccc} \mu_Y & & r \cdot \pi_Y & & \sigma_Y & & \lambda \\ \begin{bmatrix}\mu S \\ rM\end{bmatrix} & -r & \begin{bmatrix}S \\ M\end{bmatrix} & = & \begin{bmatrix}\sqrt{\nu_t}S & 0 \\ 0 & 0\end{bmatrix} & & \begin{bmatrix}\lambda_1 \\ \lambda_2\end{bmatrix}\end{array} \qquad r = r$$

- The market price of stock risk is uniquely determined, $\boxed{\lambda_1 = \frac{\mu - r}{\sqrt{\nu_t}}.}$
- The market price of volatility risk λ_2 can be chosen arbitrarily.
- The model is obviously incomplete. Thus for any given numéraire, the corresponding EMM is not unique.
- Consequently, neither the numéraire-dependent option pricing formula, nor the PDE approach deliver unique arbitrage-free option prices. They rather depend on the particular choice of λ_2.

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- Under $\mathbb{Q}$, generated by $(\lambda_1\ \lambda_2)$, the model evolves according to

$$dS_t = S_t[\boxed{r}dt + \sqrt{\nu_t}dW^{\mathbb{Q}}_{1,t}]$$

$$d\nu_t = \left[\kappa(\theta - \nu_t) - \underbrace{\lambda_{1,t}\sigma\rho\sqrt{\nu_t}}_{=(\mu-r)\sigma\rho} - \underbrace{\lambda_{2,t}\sigma\sqrt{\nu_t}\sqrt{1-\rho^2}}\right]dt$$

$$+\sigma\sqrt{\nu_t}d(\rho W^{\mathbb{Q}}_{1,t} + \sqrt{1-\rho^2}W^{\mathbb{Q}}_{2,t})$$

- Heston (1993) *chooses* $\lambda_{2,t}$ such that the drift adjustment is proportional to ν_t, i.e., $\boxed{\lambda\nu_t}$ for $\lambda \in \mathbb{R}$
- Therefore,

still the linear structure

$$d\nu_t = \left[\kappa(\theta - \nu_t) - \lambda\nu_t\right]dt + \sigma\sqrt{\nu_t}d(\rho W^{\mathbb{Q}}_{1,t} + \sqrt{1-\rho^2}W^{\mathbb{Q}}_{2,t})$$

and there is a closed-form solution for the call option price for every particular choice of $\lambda \in \mathbb{R}$.

Problem: Set up the Pricing PDE

$$\frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} r S + \frac{\partial \pi_C}{\partial v} \kappa^{\mathbb{Q}} (\theta^{\mathbb{Q}} - v)$$

$$+ \frac{1}{2} \frac{\partial^2 \pi_C}{\partial S^2} S^2 v + \frac{1}{2} \frac{\partial^2 \pi_C}{\partial v^2} \sigma^2 v$$

$$+ \frac{\partial^2 \pi_C}{\partial S \partial v} S \sqrt{v} \sigma \sqrt{v} \rho = r \pi_C$$

$$\pi_C(T, S, v) = F(S, v) \overset{e.g.}{=} (S - K)^+$$

Calibration vs. Estimation

- **Crucial Question:** How do we determine the market price of risk?
- Calibration and estimation are two ways of determining parameters in a financial model. The difference is:
  - estimation uses methods of statistics/econometrics to infer parameter values from observed *historical* behavior of asset prices and other relevant quantities
  - calibration sets parameter values so as to generate a close match between derivative prices obtained from the model and prices observed *currently* in the market.
- Estimation comes with standard errors, significance tests, and so on; analogous quantities that may serve as warning signals are not produced by calibration.
- Estimation works with models that are written under $\mathbb{P}$ (real-world measure); calibration can be applied to models that are written under $\mathbb{Q}_N$ (martingale measure corresponding to a chosen numéraire).

In our Situation

- Estimation helps us to figure out the parameters under $\mathbb{P}$

$$
\begin{aligned}
\mathrm{d}M_t &= M_t r \mathrm{d}t \\
\mathrm{d}S_t &= S_t[\mu \mathrm{d}t + \sqrt{\nu_t}\mathrm{d}W_{1,t}] \\
\mathrm{d}\nu_t &= \kappa(\theta - \nu_t)\mathrm{d}t + \sigma\sqrt{\nu_t}\mathrm{d}(\rho W_{1,t} + \sqrt{1-\rho^2}W_{2,t})
\end{aligned}
$$

- However, for pricing purposes, we need the $\mathbb{Q}$-dynamics.
- **Idea:** *Calibrate* the relevant parameters under $\mathbb{Q}$ (in particular λ) such that the model closely matches the prices of plain vanilla options.
- Use the calibrated parameters to determine arbitrage-free prices of more complicated products.

Recipe for Calibration

- Determine a closed-form solution for option prices that depends on the particular choice of the market price of risk, i.e., an expression
$$C(S_0, \nu_0, \Theta, K, T)$$
for a strike price K, time horizon T, and parameter set $\Theta = (\kappa, \theta, \sigma, \rho, \lambda)$.
- Observe market prices of options $C_1(K_1, T_1), \dots C_N(K_N, T_N)$ for various combinations of K and T.
- Solve the following minimization problem for a set of weights w:
$$\Theta^* = \arg\min_{\Theta} \sum_{i=1}^{N} w_i \left[C(S_0, \Theta, K_i, T_i) - C_i(K_i, T_i) \right]^2$$
- This shows a potential conflict between estimation and calibration: time series information can be used to determine the parameters κ and σ in the model under $\mathbb{Q}$, and these values might differ from those obtained by calibration.

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not relevant for examination.

Costs and Dividends

- In the theory we assume that assets are self-financing, but, in reality, stocks often generate dividends, and commodities typically bring storage costs.
- Strategy: specify where the dividends go (or where the costs are financed from). In this way, the given asset becomes part of a self-financing portfolio. Then derive the distribution of the asset under a suitable EMM.
- To illustrate, suppose that S_t is the price at time t of a dividend-paying stock, and assume for convenience that dividend is paid continuously at a fixed rate, as a percentage of the stock price. We show two implementations of the strategy above.

Motivation from Discrete Time

- Usual BS model:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$$
$$\mathrm{d}M_t = r M_t \,\mathrm{d}t$$

  but now suppose that the stock pays continuously a fixed-percentage dividend, i.e., the dividend received from one unit of the stock during the instantaneous interval from t to $t + \mathrm{d}t$ is $qS_t\mathrm{d}t$ where q is a constant.

- We can choose to re-invest the dividends into the stock. Let V_t be the value at time t of the portfolio that is created in this way. We have for small Δt:

$$V_{t+\Delta t} = V_t + \frac{V_t}{S_t}\left(S_{t+\Delta t} - S_t\right) + \frac{V_t}{S_t}\, qS_t\Delta t.$$

Dividends in Continuous Time

- In continuous time:

$$\mathrm{d}V_t = \frac{V_t}{S_t}\left(\mathrm{d}S_t + qS_t\,\mathrm{d}t\right) = (\mu + q)V_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t.$$

- The portfolio V_t is self-financing, so under $\mathbb{Q}$:

$$\mathrm{d}V_t = rV_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t^{\mathbb{Q}}.$$

- From $\mathrm{d}V_t = (V_t/S_t)(\mathrm{d}S_t + qS_t\,\mathrm{d}t)$ it follows that $\mathrm{d}S_t = (S_t/V_t)(\mathrm{d}V_t - qV_t\,\mathrm{d}t)$.
- Therefore

$$\mathrm{d}S_t = (r - q)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}.$$

This allows us to price options that are stated in terms of S_t.

- Alternative approach: assume that the dividends are placed in an savings account A.
- We have for small time interval of length Δt:

$$A_{t+\Delta t} = A_t + rA_t\Delta t + qS_t\Delta t$$

  so that $\mathrm{d}A_t = (rA_t + qS_t)\,\mathrm{d}t$.
- The portfolio $V_t := S_t + A_t$ is self-financing. So, under $\mathbb{Q}$,

$$\mathrm{d}V_t = rV_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}.$$

- From $\mathrm{d}V_t = \mathrm{d}S_t + \mathrm{d}A_t$ it follows that $\mathrm{d}S_t = \mathrm{d}V_t - \mathrm{d}A_t$.

- Therefore,

$$\begin{aligned} \mathrm{d}S_t &= rV_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t^{\mathbb{Q}} - (rA_t + qS_t)\,\mathrm{d}t \\ &= r(S_t + A_t)\,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t^{\mathbb{Q}} - (rA_t + qS_t)\,\mathrm{d}t \\ &= (r - q)S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t^{\mathbb{Q}}. \end{aligned}$$

  We find the same SDE for S_t under $\mathbb{Q}$ as was found on the basis of the reinvestment strategy.

- The pricing formula for a call option written on S_t becomes

$$C_0 = e^{-qT} S_0(d_1) - e^{-rT} K(d_2)$$

$$d_1 = \frac{\log(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

General Setting

- Consider an extension of the generic state space model

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\ Y_t &= \pi_Y(t, X_t). \end{aligned}$$

  by introducing an m-dimensional *dividend process* $D_t = D(t, X_t)$ representing the cumulative dividends of the m assets.
- $\mathrm{d}D_t$ represents the dividends at time t.
- The *gains process* is defined as

$$G_t = Y_t + D_t$$

- A process ϕ is called a self-financing trading strategy if

$$V_t = \phi_t' Y_t, \qquad \begin{aligned} \mathrm{d}V_t &= \phi_t' \mathrm{d}G_t \\ &= \phi_t' \mathrm{d}Y_t + \phi_t' \mathrm{d}D_t \end{aligned}$$

Discounted Gain Process

- Given a pricing kernel K, we define the deflated price process by $Y^K = KY$.
- What is an appropriate definition for the deflated gains process? $\Longrightarrow$ With dividends, it does not make sense to define the deflated gains process by $G^K = KY + KD$, since it does not take the timing and reinvestment of the dividends into account.
- Instead, we define the deflated gains process G^K s.t. deflated wealth $V^K = KV^\phi$ generated by self-financing trading strategy ϕ equals wealth generated by this trading strategy and deflated prices and gains:

$$V^K = \phi'(KY), \qquad \mathrm{d}V^K = \phi' \mathrm{d}G^K, \qquad G^K \text{ is a } \mathbb{P}\text{-martingale}$$

- We already know $Y^K = KY$. What's about D^K?

- **Easiest Formulation** (dividends are locally risk-free):

$$\mathrm{d}D_t = \mu_D(t, X_t)\mathrm{d}t$$

Then, the discounted dividends follow (check!):

$$\mathrm{d}D_t^K = K_t \mu_D(t, X_t)\mathrm{d}t$$

- **General Case** (dividends my be driven by systematic or idiosyncratic shocks):

$$\mathrm{d}D_t = \mu_D(t, X_t)\mathrm{d}t + \sigma_D(t, X_t)\mathrm{d}W_t$$

Then, the discounted dividends follow (check!):

$$\mathrm{d}D_t^K = \left[K_t \mu_D(t, X_t) + \sigma_K' \sigma_D\right]\mathrm{d}t + K_t \sigma_D' \mathrm{d}W_t$$

First FTAP with Dividends

- Given: joint process of asset prices $(Y_t)_{t\geq 0}$, cumulative dividends $(D_t)_{t\geq 0}$
- The deflated gains process G^K is given by

$$\mathrm{d}G^K = \mathrm{d}(KY) + \mathrm{d}D_t^K.$$

First Fundamental Theorem of Asset Pricing

The following are equivalent:

1. The market is free of arbitrage.
2. There is a positive adapted scalar process $(K_t)_{t\geq 0}$ such that the process $(G_t^K)_{t\geq 0}$ is a martingale under $\mathbb{P}$.

Pricing with Dividends

- By definition $K_0 = 1$, and $D_0 = 0$.
- FTAP with dividends implies:

$$G_t^K = \mathbb{E}_t[G_T^K] \qquad \Longleftrightarrow \qquad Y_t = Y_t^K + D_t^K = \mathbb{E}_t[Y_T K_T + D_T^K],$$

  in particular, $Y_0 = \mathbb{E}[Y_T K_T + D_T^K]$
- **Remark:** The FTAP works for other numéraire-measure-combinations as well. In particular, for $N_t = M_t$:

$$Y_t = \mathbb{E}_t^{\mathbb{Q}}\Big[\frac{Y_T}{M_T} + \int_t^T \frac{1}{M_u} \mathrm{d}D_u\Big]$$

- If dividends follow the dynamics $\mathrm{d}D_t = \mu_D(t, X_t)\,\mathrm{d}t$, then

$$Y_t = \mathbb{E}_t^{\mathbb{Q}}\Big[Y_T\, \mathrm{e}^{-\int_t^T r_s ds} + \int_t^T \mathrm{e}^{-\int_t^u r_s ds} \mu_D(u, X_u)\mathrm{d}u\Big],$$

  i.e., prices have a Feynman-Kac representation.