Valuation and Risk Management

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Tilburg School of Economics and Management
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Course Information

www.christoph-hambel.de

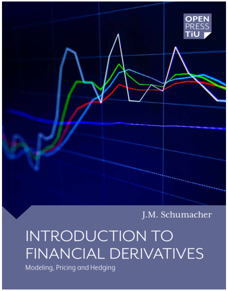
- Lecturers:
  - Christoph Hambel (financial modeling and derivative pricing)
  - Nikolaus Schweizer (numerical methods and risk measures)
  - Henk Keffert (K.R.F.Keffert@tilburguniversity.edu)
- This course ...
  - ... provides an introduction to financial modeling, pricing, and risk management beyond the Black-Scholes framework
  - ... requires some knowledge from mathematics and finance, especially from stochastic calculus (Wiener process, Itô's Lemma, Change of measure, Girsanov's Theorem, ...)
  - ... contains a guest lecture by (tba)
- Grading:
  - Exam 70%
  - Two Assignments (15% each)

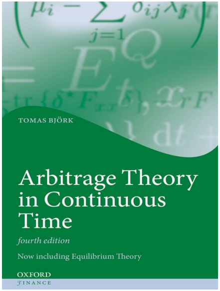
What to expect?

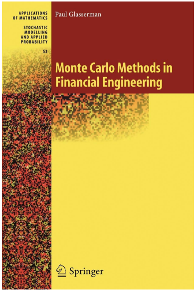
- What can you expect from us? We will...
  - ... timely provide the learning material on Canvas
  - ... also upload the slides with hand-written complements (some slides are intentionally blank)
  - ... illustrate the lecture by examples
  - ... provide problem sets and a sample exam to practice the material
  - ... be available for questions
  - ... offer a virtual Q&A session after the last lecture
- What will we expect from you? You should ...
  - ... be well-prepared when you come to the lecture
  - ... actively participate in the lecture
  - ... take the opportunity and ask us questions during the classes

Recommended Literature

- We do not make any book the *mandatory* reading for this course. However, we highly recommend the following textbooks:
  - Schumacher, J.M.: *Introduction to Financial Derivatives: Modeling, Pricing and Hedging* (Open Press TiU)
  - Björk, T.: *Arbitrage Theory in Continuous Time (Oxford)*
  - Glasserman, P.: *Monte-Carlo Methods in Financial Engineering* (Springer)
- This course follows the notation in Schumacher (2020), which contains a lot of exercises.







Preliminary Schedule

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Please notice that the plan can change!

- Mon, 28.08.2023, 12:45, WZ105 slides 1 - 24.
- Mon, 04.09.2023, 12:45, WZ105
- Tue, 05.09.2023, 14:45, CUBE 218
- Mon, 11.09.2023, 12:45, WZ105
- Mon, 18.09.2023, 12:45, WZ105
- Tue, 19.09.2023, 14:45, CUBE 218
- Mon, 25.09.2023, 12:45, WZ105
- Mon, 02.10.2023, 12:45, WZ105
- Tue, 03.10.2023, 14:45, CUBE 218
- Mon, 09.10.2023, 12:45, WZ105
- Tue, 10.10.2023, 14:45, CUBE 218

Structure of the Course (First Half)

1. Introduction to Financial Modeling
   - Discrete vs. Continuous Time Modeling
   - Fundamentals from Stochastic Calculus
2. Continuous time: Generic State Space Model
   - Framework
   - No Arbitrage and the First FTAP
   - The Numéraire-dependent Pricing Formula
   - Replication and the Second FTAP
   - The PDE Approach
3. Contingent Claim Pricing
   - Black-Scholes Revisited
   - Option Pricing in Incomplete Markets
   - Models with Dividends

Structure of the Course (First Half)

4. Fixed Income Modeling
   - Bonds and Yields
   - Interest Rates and Interest Rate Derivatives
   - Short Rate Models for the TSIR
   - Empirical Models
   - The Heath-Jarrow-Morton Framework
   - LIBOR Market Model and Option Pricing
5. A Brief Introduction to Credit Risk
   - Reduced-Form Modeling
   - Merton's Firm Value Model

Part I

Introduction to Financial Modeling

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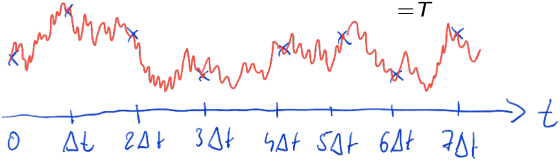
- Discrete time with time horizon T:

$$t \in \{0, \Delta t, 2\Delta t, \ldots, (n-1)\Delta t, \underbrace{n\Delta t}_{=T}\} = \{i\Delta t \mid i = 0, \ldots, n\}$$



- Continuous time as a limit of discrete time ($\Delta t \to 0$ as $n \to \infty$):

$$t \in [0, T]$$

Modeling in Discrete Time: First Idea

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- Risk-free asset (bond) paying a constant interest rate:

$$B_{t+\Delta t} = B_t(1 + r \cdot \Delta t) \quad \Longleftrightarrow \quad \frac{\Delta B_{t+\Delta t}}{B_t} = r \cdot \Delta t$$

- Risky asset (stock):

$$S_{t+\Delta t} = S_t(1 + \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}), \qquad \nu_{t+\Delta t} \sim_{i.i.d.} (0, 1)$$

volatility (σ)

expected rate of return (μ)

- Return:

$\mu > r$

$\mu - r$: risk premium

$$\frac{\Delta S_{t+\Delta t}}{S_t} = \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

Problem: Returns are not necessarily bounded from below by -1 and thus asset prices can be negative.

Log Returns

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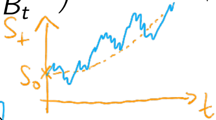
- Way out? $\rightarrow$ Model log returns, L_t, and take the exponential:

$$S_{t+\Delta t} = S_t \mathrm{e}^{\Delta L_{t+\Delta t}}$$

- Risk-free asset (bond):

$$B_{t+\Delta t} = B_t \mathrm{e}^{r \cdot \Delta t} \quad \Longleftrightarrow \quad r\Delta t = \ln\left(\frac{B_{t+\Delta t}}{B_t}\right) = \Delta \ln B_{t+\Delta t}$$



- Risky asset (stock):

$$\Delta L_{t+\Delta t} = \ln(S_{t+\Delta t}) - \ln(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

- Now, we take the limit to continuous time, i.e., we increase the number of periods ($n \rightarrow \infty$) while keeping the time horizon constant, i.e., $\Delta t = \frac{T}{n} \rightarrow 0$.

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$$S_T = S_0 \prod_{i=0}^{n-1} \mathrm{e}^{\Delta L_{(i+1)\Delta t}}$$

$$= S_0 \exp\left\{ \sum_{i=0}^{n-1} \left[\left(\mu - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma \cdot \nu_{(i+1)\Delta t} \cdot \sqrt{\Delta t} \right] \right\}$$

$n \cdot \Delta t = T$

$$= S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{\Delta t} \cdot \sum_{i=1}^{n} \nu_{i\Delta t} \right\}$$

$$= S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{T} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i\Delta t} \right\}$$

According to the CLT: $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i\Delta t} \to_d Z_T \sim \mathcal{N}(0,1)$ as $n \to \infty$, i.e.,

$$S_T \to_d S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{T} \cdot Z_T \right\}$$

From Discrete to Continuous Time

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- In the limit, the log return is normally distributed:

$$L_T = L_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma \cdot \sqrt{T} \cdot Z_T$$

- Consequently, in the limit S_T is log-normally distributed with

$$\text{mean:} \qquad \mathbb{E}[S_T] = S_0 e^{\mu \cdot T}$$

$$\text{variance:} \quad \text{var}(S_T) = S_0^2 e^{2\mu \cdot T}[e^{\sigma^2 T} - 1]$$

- Does this mean that any discrete-time model converges to a log-normal distribution? No! But we need non-iid shock s.t. the CLT cannot be applied.
- How can we model asset prices in continuous time?

$\Delta S_{t+\Delta t} = S_t [\mu \Delta t + \sigma \varepsilon_{t+\Delta t} \sqrt{\Delta t}]$

↳ replace by $dS_t = S_t [\mu dt + \sigma dW_t]$

Trading in Discrete Time

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- Assume that there is a frictionless financial market (i.e., no taxes, no transaction costs, no short-selling constraints, ...)
- Throughout the lecture we will be using vector notation:

$Y_t = \begin{pmatrix} M_t \\ S_t \end{pmatrix}$

m : number of basic assets

$\varphi_t = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Y_t : m-dimensional vector of asset prices at time t

ϕ_t : vector of number of units of assets held at time t

$V_t = 2M_t + 1S_t$

- Portfolio value generated by the *portfolio strategy* (or *trading strategy*) ϕ:

$$V_t = \phi_t' Y_t. = \sum_{i=1}^{m} \phi_{it} Y_{it}$$

- A portfolio strategy ϕ is *self-financing* if trading neither generates nor destroys money, i.e.,

$$\phi_{t-\Delta t}' Y_t = \phi_t' Y_t.$$

Trading does not change the portfolio value

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- Suppose that rebalancing takes place at times $0 < t_1 < \cdots < t_n = T$, i.e., $t_j = j\Delta t$.

$V_{t_{j+1}} = \phi'_{t_{j+1}} Y_{t_{j+1}} = \phi'_{t_j} Y_{t_{j+1}}$

$$
\begin{aligned}
V_T &= V_0 + \sum_{j=0}^{n-1} (V_{t_{j+1}} - V_{t_j}) && \text{(telescope rule)} \\
&= V_0 + \sum_{j=0}^{n-1} \phi'_{t_j} (Y_{t_{j+1}} - Y_{t_j}) && \text{(self-financing portfolio)} \\
&= V_0 + \sum_{j=0}^{n-1} \phi'_{t_j} \, \Delta Y_{t_{j+1}}.
\end{aligned}
$$

$\xrightarrow[\Delta t \to 0]{n \to \infty} V_0 + \int_0^T \phi'_t \, dY_t$

accumulated gains generated by your portfolio

- The sum $\sum_{j=0}^{n-1} \phi'_{t_j} \, \Delta Y_{t_{j+1}}$ converges in some sense to the stochastic integral $\int_0^T \phi'_t \, \mathrm{d} Y_t$ even if the integrator is of infinite variation.
- The continuous-time version of self-financing is $V_T = V_0 + \int_0^T \phi'_t \, \mathrm{d} Y_t$.

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- We need adequate tools for modeling asset prices in continuous time that can be interpreted along the lines of

$$(1) \quad \frac{\Delta B_{t+\Delta t}}{B_t} = r \cdot \Delta t$$

$$(2) \quad \frac{\Delta S_{t+\Delta t}}{S_t} = \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

  and that preserve the limit distribution of the stock return.
- Replace (1) by an ODE and (2) by an SDE:

$$(1') \quad \frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t$$

$$(2') \quad \frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

- Replace the self-financing condition $\phi'_{t-\Delta t} Y_t = \phi'_t Y_t$ by $V_T = V_0 + \int_0^T \phi'_t \, \mathrm{d}Y_t$ for an adequately defined stochastic integral.

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Stochastic Processes

- Consider a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$
  - Ω denotes the state space.
  - $\mathcal{A} \subset 2^\Omega$ denotes a sigma algebra that contains all events for which probabilities can be assigned.
  - $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration, which models the set of information available at time t.
  - $\mathbb{P} : \mathcal{A} \to [0,1]$ is a probability measure, which we refer to as *real-world* probability measure.
- A stochastic process X is a collection of random variables $(X_t)_{t\geq 0}$ indexed by time.
- **Remarks:**
  - Throughout the course, we assume that all processes are continuous (i.e., "no jumps" a.s.) and adapted (i.e., "realization X_t is known at time t"). Formulas become more involved if we relax this assumption.
  - I will avoid technical terms (e.g., measurability, integrability), but focus on economic interpretations. I will rather assume that all processes satisfy all relevant conditions.

Brownian Motion

Definition (Brownian Motion)

A one-dimensional (standard) *Brownian motion* (aka *Wiener Process*) is a stochastic process $W = (W_t)_{t \geq 0}$ such that $W_0 = 0$ a.s. and

- $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$ (*stationary increments*).
- $W_t - W_s$ is independent of $W_u - W_v$ for $0 \leq v < u \leq s < t$ (*independent increments*).

- A k-dimensional standard Brownian motion $W = (W_1, \ldots, W_k)$ is a k-dimensional vector of independent Brownian motions.
- Notice that the paths of a Brownian motion are continuous (a.s.) but nowhere differentiable. In particular, the paths of Brownian motion have infinite length on any interval ("infinite variation").

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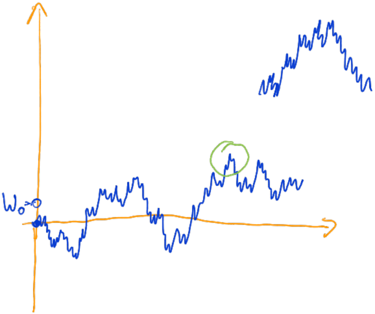

$$\lim_{n\to\infty} \sum_{i=1}^{n} |W_{t_{j+1}} - W_{t_j}| = \infty$$

Arclength of any path on any interval is infinite

$[0, T]$, $n\Delta t = T$

$$\lim_{n\to\infty} \sum_{i=1}^{n} (W_{t_{j+1}} - W_{t_j})^2 = T$$

$$Var(W_T) = T$$

Martingales

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Definition (Martingale)

A stochastic process $Z = (Z_t)_{t \geq 0}$ is said to be a *martingale* if "the best estimate of the future is the present", i.e.,

$$E_s[Z_t] = Z_s \qquad t \geq s$$

- Martingales relate to "fair games" and are often thought of as "purely stochastic" processes, that is, containing no trend or being constant in expectation..
- Example: Brownian motion is a martingale.
- There are many generalizations of martingales, e.g.,
  - Submartingales ("non-decreasing in expectation")
  - Supermartingales ("non-increasing in expectation")
  - Local martingales ("if stopped process is a martingale")
  - Semimartingales ("local martingale + process of finite variation")

Itô Integral

- The stochastic integral (a.k.a. Itô integral) is defined by

$$\int_0^T X_t \, \mathrm{d}Z_t = \lim_{n \to \infty} \sum_{j=0}^{n} X_{t_j} (Z_{t_{j+1}} - Z_{t_j})$$

  where Z is a semimartingale, X is an adapted process, and the stochastic limit is taken in the sense of refining partitions (i.e., intermediate points $t_0, t_1, \ldots, t_n$ become more and more dense on the interval $[0, T]$ as n tends to infinity).
- The construction of the limit and prove of convergence is not trivial, since in general the integrator is of infinite variation.
- Such a limit does not necessarily exist pathwise.
- Note: by contrast to the Riemann-Stieltjes integral, the integrand is evaluated at the left end t_j.
- The stochastic integral is itself a random variable.

Stochastic Differential Equation

$\int_0^t dX = X_t - X_0$

Definition (Stochastic Differential Equation)

Let W be a standard Brownian motion. An expression of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

for given functions $\mu(t, X_t)$ (*drift*) and $\sigma(t, X_t)$ (*volatility*) is called a *stochastic differential equation* (SDE) driven by Brownian motion and should be understood as a short-hand notation for the integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s.$$

- If the drift $\mu(t, X_t)$ is zero, then the solution is a martingale.
- This definition can be generalized to SDEs driven by jump processes (e.g., Poisson processes).

Quadratic (Co-)Variation

- Let X, Y be two real-valued stochastic processes, then their *quadratic covariation process* is defined as

$$[X, Y]_t = \lim_{\Delta t \to 0} \sum_{j=0}^{t} (X_{t_{j+1}} - X_j)(Y_{t_{j+1}} - Y_j)$$

- The *quadratic variation process* of X is defined by

$$[aY, bX] = a \cdot b \cdot [X, Y]$$

$$[X]_t = [X, X]_t$$

- Rules for quadratic (co)-variation:
  - linearity in both arguments
  - $[X, g] = 0$ if g is a continuous function of bounded variation
  - $\mathrm{d}[W_1, W_2] = \rho\,\mathrm{d}t$ for BMs with correlation coefficient ρ; $\mathrm{d}[W] = \mathrm{d}t$
  - if $\mathrm{d}X = \mu_X\,\mathrm{d}t + \sigma_X\,\mathrm{d}W_1$ and $\mathrm{d}Y = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}W_2$, then

$$\mathrm{d}[X, Y] = \sigma_X \sigma_Y \rho\,\mathrm{d}t, \qquad \mathrm{d}[X] = \sigma_X^2\,\mathrm{d}t$$

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Theorem (Itô's Lemma for continuous semimartingales)

Let X be a continuous real-valued semimartingale, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)\,\mathrm{d}t + \frac{\partial}{\partial x} f(t, X_t)\,\mathrm{d}X_t + \underbrace{\frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t)\,\mathrm{d}[X, X]_t}_{\text{Itô-term}}.$$

Theorem (Itô's Lemma for Itô processes)

Let X be an Itô process $\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}W_t$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \left[\frac{\partial}{\partial t} f(t, X_t) + \frac{\partial}{\partial x} f(t, X_t)\mu_X + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t)\,\sigma_X^2\right]\mathrm{d}t + \frac{\partial}{\partial x} f(t, X_t)\sigma\,\mathrm{d}W_t.$$

Problem: Derive the stock price in the Black-Scholes model and show that it is strictly positive almost surely.

Solution: $dS_t = S_t(\mu\, dt + \sigma\, dW_t)$

$f(t,x) = \log(x)$

$f_t = 0, \quad f_x = \frac{1}{x}, \quad f_{xx} = -\frac{1}{x^2}$

$$d \log S_t = f_t\, dt + f_x\, dX_t + \frac{1}{2} f_{xx}\, d[X]_t$$

$$= \frac{1}{S_t} S_t (\mu\, dt + \sigma\, dW_t) + \frac{1}{2}\left(-\frac{1}{S^2}\right) S^2 \sigma^2\, dt$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW$$

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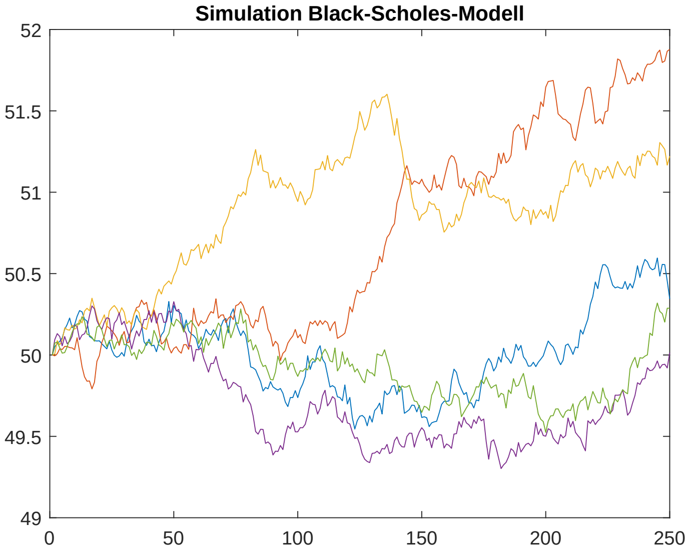
$$\Rightarrow \quad \log S_t = \log S_0 + (\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t$$

$$\boxed{S_t = S_0 \exp\{(\mu - \tfrac{1}{2}\sigma^2)t + \sigma W_t\}}$$

Geometric Brownian Motion

$\Rightarrow$ Stock price remains strictly positive if $S_0 > 0$.

Geometric Brownian Motion



Theorem (Itô's Lemma for continuous semimartingales)

Let $X = (X_t^1, \ldots, X_t^n)_{t \geq 0}$ be a continuous $\mathbb{R}^n$-valued semimartingale, and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)\, \mathrm{d}t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(t, X_t)\, \mathrm{d}X_t^i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t)\, \mathrm{d}[X^i, X^j]_t.$$

Special Case: $f(X, Y) = XY$: Itô product rule:

$$\mathrm{d}(XY)_t = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + \mathrm{d}[X, Y]_t$$

Itô's Lemma: Multivariate Version

Theorem (Itô's Lemma for multivariate Itô processes)

Let W be a k-dimensional standard Brownian motion, X be a $\mathbb{R}^n$-valued Itô process with dynamics

$$\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}W_t$$

for sufficiently smooth functions $\mu_X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$. Let $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}$-function with gradient $\nabla f(t, X_t)$ and Hessian matrix $H_f(t, X_t)$, then

$$\mathrm{d}f(t, X_t) = \Big[\underbrace{\frac{\partial}{\partial t} f(t, X_t)}_{\in\mathbb{R}} + \underbrace{\nabla f(t, X_t)}_{\in\mathbb{R}^n} \cdot \underbrace{\mu_X}_{\in\mathbb{R}^n} + \frac{1}{2}\mathrm{tr}\Big(\underbrace{H_f(t, X_t)}_{\in\mathbb{R}^{n\times n}} \underbrace{\sigma_X}_{\in\mathbb{R}^{n\times k}} \underbrace{\sigma_X'}_{\in\mathbb{R}^{k\times n}} \Big)\Big]\mathrm{d}t + \underbrace{\nabla f(t, X_t)}_{\in\mathbb{R}^n} \underbrace{\sigma_X}_{\in\mathbb{R}^{n\times k}} \underbrace{\mathrm{d}W_t}_{\in\mathbb{R}^k}$$

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$dS_t = S_t[\mu dt + \sigma dW_t]$, $dB_t = B_t r dt$

[?] What are the dynamics of $\frac{S}{B}$?

$f(x,y) = \frac{x}{y}$, $f_x = \frac{1}{y}$, $f_y = -\frac{x}{y^2}$

$f_{xx} = 0$, $f_{yy} = 2\frac{x}{y^3}$

$f_{xy} = -\frac{1}{y^2}$

$$d\left(\frac{S}{B}\right) = \frac{1}{B}dS - \frac{S}{B^2}dB$$

$$= \frac{1}{B}S(\mu dt + \sigma dW) - \frac{S}{B^2}Br dt$$

$$= \frac{S}{B}\left[(\mu - r)dt + \sigma dW\right] \overset{(*)}{=} \frac{S}{B}\left((\mu - r)dt + \sigma d\tilde{W} - \sigma\lambda dt\right)$$

Construct a measure $\mathbb{Q} \sim \mathbb{P}$ s.t. $\frac{S}{B}$ is a $\mathbb{Q}$-martingale:

Under $\mathbb{Q}$: drift must be zero

$d\tilde{W} = \lambda dt + dW \Leftrightarrow dW = d\tilde{W} - \lambda dt$ (*)

$$\Rightarrow \mu - r - \sigma\lambda = 0 \Leftrightarrow \boxed{\lambda = \frac{\mu - r}{\sigma}}$$

Change of Measure

Definition (Equivalent Probability Measure)

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be *equivalent*, $\mathbb{P} \sim \mathbb{Q}$, if both measures possess the same null sets, i.e., for all events $A \in \mathcal{A}$

$$\mathbb{P}(A) = 0 \quad \Longleftrightarrow \quad \mathbb{Q}(A) = 0.$$

- In our pricing applications, we consider equivalent probability measures that are associated to a numéraire.
- A numéraire is any self-financing portfolio ϕ that generates strictly positive wealth $V_t^{\phi} = \phi_t' Y_t$
- A probability measure $\mathbb{Q} \sim \mathbb{P}$ is said to be an equivalent martingale measure if for every asset with price process Y^i $(i = 1, \ldots, m)$ the price expressed in terms of the numéraire V_t^{ϕ} is a martingale under $\mathbb{Q}$.

Change of Measure – Radon-Nikodym Theorem

- The following theorem states how to switch between two equivalent probability measures.

Theorem (Radon-Nikodym)

Let $\mathbb{P} \sim \mathbb{Q}$ denote two equivalent probability measures, then there exists a unique (a.s.), positive random variable $\theta = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[\theta X], \qquad \mathbb{E}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{\theta}\right]$$

for all real-valued random variables X. In particular,

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[\theta\, 1_A]$$

θ is called the *Radon-Nikodym density* (or *Radon-Nikodym derivative*).

- Critical Question: How can we perform a change of measure if the market is driven by Brownian motions?

Girsanov Theorem

Theorem (Girsanov)

Suppose that a measure $\mathbb{Q}$ is defined in terms of a measure $\mathbb{P}$ by the Radon-Nikodym process $(\theta_t)_{t\geq 0}$, with

$$\mathrm{d}\theta_t = -\lambda_t \theta_t \, \mathrm{d}W_t$$

where W is a Brownian motion under $\mathbb{P}$ and λ is a continuous adapted process. Then the process $\widetilde{W}$ defined by $\widetilde{W}_0 = 0$ and

$$\mathrm{d}\widetilde{W}_t = \lambda_t \, \mathrm{d}t + \mathrm{d}W_t$$

is a Brownian motion under $\mathbb{Q}$.

This works as well for vector BMs; in this case, write

$$\mathrm{d}\theta_t = -\theta_t \lambda_t' \, \mathrm{d}W_t, \quad \mathrm{d}\widetilde{W}_t = \lambda_t \, \mathrm{d}t + \mathrm{d}W_t.$$

Some Remarks

- The stochastic differential equation $\mathrm{d}\theta_t = -\lambda_t \theta_t \, \mathrm{d}W_t$ has a unique solution, the Radon-Nikodym *process*:

$$\theta_t = \mathcal{E}(\lambda)_t = \exp\Big(-\int_0^t \lambda_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \lambda_s^2 \mathrm{d}s \Big)$$

- The process $\mathcal{E}(\lambda)$ is called the stochastic exponential or Doléans-Dade exponential of λ.
- The Radon-Nikodym *derivative* is given by

$$\theta_T = \exp\Big(-\int_0^T \lambda_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda_s^2 \mathrm{d}s \Big)$$

- The Radon-Nikodym *process* is a $\mathbb{P}$-martingale, i.e.,

$$\theta_t = \mathbb{E}_t[\theta_T].$$