No Risk: Assumptions

We first consider the case without risks under the following assumptions:

1. The number of survivors equals the number of expected survivors (the "best estimate"), i.e.,

$$N_{x+1,t+1} = N_{x,t} p_{x,t}^{BE(t)}.$$

2. The "best estimate" cohort life table of time $t+1$ follows from the "best estimate" cohort life table of time t (by excluding the column corresponding to time t), i.e.,

$$p_{x,t+1+\tau}^{BE(t+1)} = p_{x,t+1+\tau}^{BE(t)}.$$

3. Financial assets generate known returns $r_{t+\tau}$ between periods $t+\tau$ and $t+\tau+1$ with

$$r_{t+\tau} = R_{t+\tau}(t+\tau+1).$$

No Financial Risk...

- It's easy to check that without financial risk the following relation holds

$$\big(1 + R_t(t+\tau)\big)^{\tau} = \prod_{j=0}^{\tau-1}(1 + r_{j+\tau}).$$

- *Proof*:

- Consequently,

$$a_{x,t}^{BE(t)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{BE(t)} \prod_{j=0}^{\tau-1} \frac{1}{1 + r_{j+\tau}}.$$

A Recursive Relation

- We can now derive the following recursive relationship:

$$
\begin{aligned}
a_{x,t}^{BE(t)} &= \sum_{\tau=1}^{\infty} {}_{\tau}p_{x,t}^{BE(t)} \prod_{j=0}^{\tau-1} \frac{1}{1+r_{j+\tau}} \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \Big(1 + \sum_{\tau=2}^{\infty} {}_{\tau-1}p_{x+1,t+1}^{BE(t)} \prod_{j=1}^{\tau-1} \frac{1}{1+r_{j+\tau}}\Big) \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \Big(1 + \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t)} \prod_{j=0}^{\tau-1} \frac{1}{1+r_{j+1+\tau}}\Big) \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \Big(1 + \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t)} \frac{1}{1+R_{t+1}(t+1+\tau)}\Big) \\
&= p_{x,t}^{BE(t)} \frac{1}{1+r_t} \big(1 + a_{x+1,t+1}^{BE(t+1)}\big)
\end{aligned}
$$

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- Consequently,

$$a_{x,t}^{BE(t)} = p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + a_{x+1,t+1}^{BE(t+1)}\right)$$

$$a_{x+1,t+1}^{BE(t+1)} = (1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1.$$

- Therefore, the liabilities at $t+1$ are now given by

$$\begin{aligned} L_{t+1}^{BE(t+1)} &= \sum_{x \in \mathcal{X}} N_{x+1,t+1} a_{x+1,t+1}^{BE(t+1)} \\ &= \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)} \left((1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1 \right) \\ &= (1+r_t) L_t^{BE(t)} - \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}. \end{aligned}$$

- The pension fund invests its assets at the capital market rate yielding a return r_t. Therefore, $\widetilde{A}_{t+1} = A_t(1 + r_t)$.
- Using these assets, the fund pays off the first unit of the annuities to all surviving members of the fund at time $t + 1$. The total payoff equals

$$\sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}.$$

- Thus, the resulting value of the assets at time $t + 1$ is given by

$$A_{t+1} = A_t(1 + r_t) - \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}.$$

- The funding ratio at time $t+1$ is thus

$$FR_{t+1}^{BE(t+1)} = \frac{A_{t+1}}{L_{t+1}^{BE(t+1)}} = \frac{A_t(1+r_t) - \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}}{L_t^{BE(t)}(1+r_t) - \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t}^{BE(t)}}.$$

- Consequently, if
  - $FR_t^{BE(t)} < 1$: the funding ratio declines over time.
  - $FR_t^{BE(t)} = 1$: the funding ratio stays stable over time.
  - $FR_t^{BE(t)} > 1$: the funding ratio increases over time.

Micro Longevity Risk: Assumptions

- We adapt the following no longevity risk assumption (Assumption 1, Slide 155):

$$N_{x+1,t+1} = N_{x,t} p_{x,t}^{BE(t)}.$$

- Instead, we assume that survival probabilities are known in advance although the remaining lifetime is uncertain. We assume that the number of survivors after one period follows a binomial distribution:

$$N_{x+1,t+1}^{(g)} \sim \mathcal{B}(N_{x,t}^{(g)}, \, p_{x,t}^{(g)}) \qquad \forall x \in \mathcal{X}.$$

- The "best estimate" cohort life table of time $t+1$ follows from the "best estimate" cohort life table of time t (by excluding the column corresponding to time t), i.e.,

$$p_{x,t+1+\tau}^{BE(t+1)} = p_{x,t+1+\tau}^{BE(t)}.$$

No Interest Rate Risk

- For the moment, we stick to the assumption that financial assets generate known returns $r_{t+\tau}$ between periods $t+\tau$ and $t+\tau+1$ with

$$r_{t+\tau} = R_{t+\tau}(t+\tau+1).$$

- Under these assumptions, the recursive relationship for the price of an annuity contract remains valid, i.e.,

$$a_{x,t}^{BE(t)} = p_{x,t}^{BE(t)} \frac{1}{1+r_t} \left(1 + a_{x+1,t+1}^{BE(t+1)}\right),$$

$$a_{x+1,t+1}^{BE(t+1)} = (1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1.$$

Assets, Liabilities, and Funding Ratio

- We can now calculate the one-period-ahead assets, liabilities, and the funding ratio:
- Assets (same calculations as before):

$$A_{t+1} = A_t(1+r_t) - \sum_{x \in \mathcal{X}} \underbrace{N_{x+1,t+1}}_{\neq N_{x,t} p_{x,t}^{BE(t)}}$$

- Liabilities (same calculations as before):

$$L_{t+1}^{BE(t+1)} = \sum_{x \in \mathcal{X}} N_{x+1,t+1} \left((1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1 \right)$$

- Funding ratio:

$$FR_{t+1} = \frac{A_{t+1}}{L_{t+1}^{BE(t+1)}} = \frac{A_t(1+r_t) - \sum_{x \in \mathcal{X}} N_{x+1,t+1}}{\sum_{x \in \mathcal{X}} N_{x+1,t+1} \left((1+r_t) \frac{a_{x,t}^{BE(t)}}{p_{x,t}^{BE(t)}} - 1 \right)}$$

Illustration

- We illustrate micro longevity risk using the following stylized example.
- We consider one age group ($x = 65$) at one specific year (namely, $t = 2019$).
- We present results for males and females separately.
- We generate 10,000 scenarios.
- We consider $N_{65,2019} = 1{,}000$, 10,000, and 50,000.
- We assume $r_{t+\tau} = 0$ for all $\tau \geq 0$.
- For the life table at time t, we use the Lee-Carter best estimate, based on the sample of Dutch males and females 1970-2019.
  - Males: $q_{65,2019}^{BE(2019)} = 1.07\%$, $p_{65,2019}^{BE(2019)} = 98.93\%$, $a_{65,2019}^{BE(2019)} = 18.95$.
  - Females: $q_{65,2019}^{BE(2019)} = 0.76\%$, $p_{65,2019}^{BE(2019)} = 99.24\%$, $a_{65,2019}^{BE(2019)} = 21.96$.

Implementation in Matlab

- To simulate a sample from a binomial distribution with parameters n and p, you could make use of the command

  `binornd(n,p)`,

  where $n = N_{65,2019}$ and $p = p_{65,2019}^{BE(2019)}$.

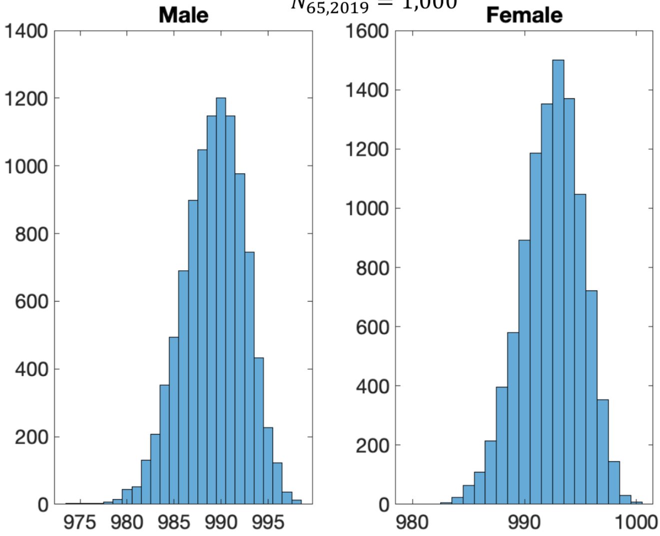
- To make a histogram of the data in the vector y (e.g., simulated funding ratio), you could make use of the command

  `histogram(y,edges)`,

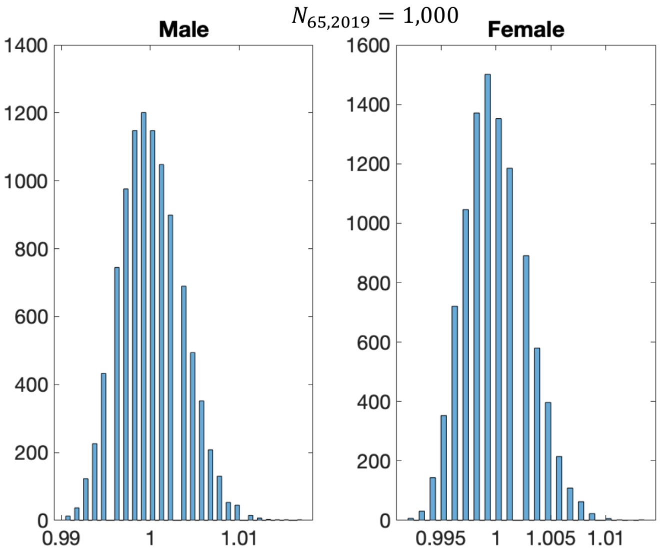
  where `edges` is a vector with the edges of the histogram columns.

Histogram Survivors

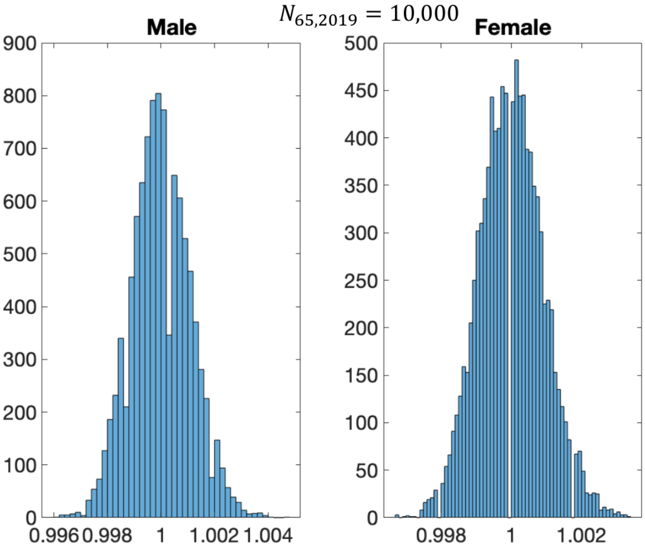
$N_{65,2019} = 1{,}000$



Histogram Funding Ratio

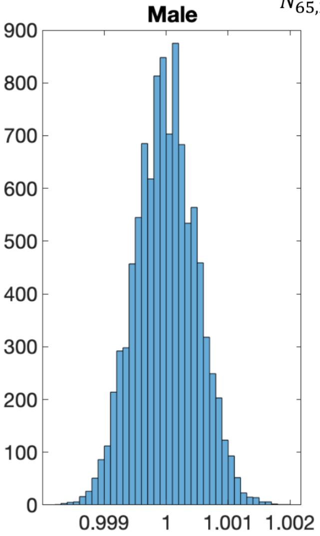


Histogram Funding Ratio



Histogram Funding Ratio

$N_{65,2019} = 50{,}000$



Macro Longevity Risk: Assumptions

- We adapt the following no macro longevity risk assumption (Assumption 2, Slide 155):

$$p_{x,t+1+\tau}^{BE(t+1)} = p_{x,t+1+\tau}^{BE(t)}$$

- Instead of assuming that the best estimate cohort life table of time $t+1$ follows from the best estimate cohort life table of time t, we follow a model-based approach.

- There are many ways in which Macro Longevity Risk can be quantified.

- We choose and illustrate one specific way, focusing on the Lee-Carter approach.

- Recall that the best estimate survival probabilities are constructed as follows:

$$p_{x,t+\tau}^{BE(t)} = \exp\left(-\widehat{m}_{x,t+\tau}^{BE(t)}\right),$$

- where

$$\begin{aligned}\widehat{m}_{x,t+\tau}^{BE(t)} &= \exp\left(\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)}\widehat{\kappa}_{t+\tau}^{BE(t)}\right),\\ \widehat{\kappa}_{t+\tau}^{BE(t)} &= \widehat{\kappa}_t^{(t)} + \tau\widehat{c}^{(t)}\end{aligned}$$

- The superindex t in case of $\widehat{\alpha}_x^{(t)}, \widehat{\beta}_x^{(t)}, \widehat{c}^{(t)}, \widehat{\kappa}_t^{(t)}$ refers to the sample based on which these parameters have been estimated, namely, the sample with data from time $t_0 = t - T$ to $t_0 + T = t$.

Macro Longevity Risk: Simplifying Assumptions

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- To keep the setting simple (although the notation is already cumbersome), we make the following simplifying assumption:
- Macro longevity risk only due to change in $\widehat{\kappa}_{t+1}^{(t+1)}$ compared to the best estimate $\widehat{\kappa}_{t+1}^{BE(t)} = \widehat{\kappa}_{t}^{(t)} + \widehat{c}^{(t)}$:

$$\widehat{\kappa}_{t+1}^{(t+1)} = \widehat{\kappa}_{t}^{(t)} + \widehat{c}^{(t)} + \delta_{t+1}.$$

- We then (re-)estimate $\widehat{c}^{(t+1)}$, now using the sample with data from time $t_0 = t - T$ to $t_0 + T + 1 = t + 1$:

$$\widehat{c}^{(t)} = \frac{\widehat{\kappa}_{t}^{(t)} - \widehat{\kappa}_{t-T}^{(t)}}{T-1} \quad \longrightarrow \quad \widehat{c}^{(t+1)} = \frac{\widehat{\kappa}_{t+1}^{(t+1)} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T}$$

- No macro longevity risk due to a possible change in $\widehat{\alpha}_{x}^{(t+1)}$ or $\widehat{\beta}_{x}^{(t+1)}$.

- Recall: $p_{x,t+\tau}^{BE(t)} = \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \widehat{\kappa}_{t+\tau}^{BE(t)}}}$
- Therefore,

$$
\begin{aligned}
p_{x,t+1+\tau}^{BE(t+1)} &= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t+1)} + \widehat{\beta}_x^{(t+1)} \widehat{\kappa}_{t+1+\tau}^{BE(t+1)}}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_{t+1}^{(t+1)} + \tau \widehat{c}^{(t+1)} \right)}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1} + \tau \frac{\widehat{\kappa}_{t+1}^{(t+1)} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T} \right)}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1} + \tau \frac{\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \delta_{t+1} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T} \right)}} \\
&= \mathrm{e}^{-\mathrm{e}^{\widehat{\alpha}_x^{(t)} + \widehat{\beta}_x^{(t)} \left(\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} + \tau \frac{\widehat{\kappa}_t^{(t)} + \widehat{c}^{(t)} - \widehat{\kappa}_{t+1-(T+1)}^{(t+1)}}{T} + \delta_{t+1} (1 + \frac{\tau}{T}) \right)}}
\end{aligned}
$$

- Maintaining assumptions 1 and 3 from slide 155, we can now calculate the evolution of assets and liabilities.
- Assets (same calculations as before):

$$A_{t+1} = A_t(1+r_t) - \sum_{x\in\mathcal{X}} \underbrace{N_{x+1,t+1}}_{=N_{x,t}p_{x,t}}$$

$$= (1+r_t)\sum_{x\in\mathcal{X}} N_{x,t}a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}$$

- Liabilities (same calculations as before):

$$L_{t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x+1,t+1}a_{x+1,t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}a_{x+1,t+1}^{BE(t+1)}$$

where

$$a_{x+1,t+1}^{BE(t+1)} = \sum_{\tau=1}^{\infty} {}_{\tau}p_{x+1,t+1}^{BE(t+1)} \prod_{j=0}^{\tau-1} \frac{1}{1+r_{t+1+j}}.$$

- Consequently, the funding ratio becomes

$$FR_{t+1} = \frac{(1 + r_t) \sum_{x \in \mathcal{X}} N_{x,t} a_{x,t}^{BE(t)} - \sum_{x \in \mathcal{X}} N_{x,t} p_{x,t}}{\sum_{x \in \mathcal{X}} N_{x,t} p_{x,t} a_{x+1,t+1}^{BE(t+1)}}.$$

- To illustrate the effect of macro longevity risk, we consider only one age group:

$$FR_{t+1} = \frac{(1 + r_t) a_{x,t}^{BE(t)} - p_{x,t}}{p_{x,t} a_{x+1,t+1}^{BE(t+1)}}$$

Illustration

- We illustrate macro longevity risk using the following stylized example.
- We consider one age group ($x = 65$) at one specific year (namely, $t = 2019$).
- We present results for males and females separately.
- We generate 10,000 scenarios.
- We assume $r_{t+\tau} = 0$ for all $\tau \geq 0$.
- We assume $\delta_{t+1} \sim \mathcal{N}\left(0, (\sigma_\delta^{(g)})^2\right)$ with $\sigma_\delta^{(m)} = 2.8653$, $\sigma_\delta^{(f)} = 3.4771$.
- For the life table at time t, we use the Lee-Carter best estimate, based on the sample of Dutch males and females 1970-2019.
  - Males: $q_{65,2019}^{BE(2019)} = 1.07\%$, $p_{65,2019}^{BE(2019)} = 98.93\%$, $a_{65,2019}^{BE(2019)} = 18.95$.
  - Females: $q_{65,2019}^{BE(2019)} = 0.76\%$, $p_{65,2019}^{BE(2019)} = 99.24\%$, $a_{65,2019}^{BE(2019)} = 21.96$.

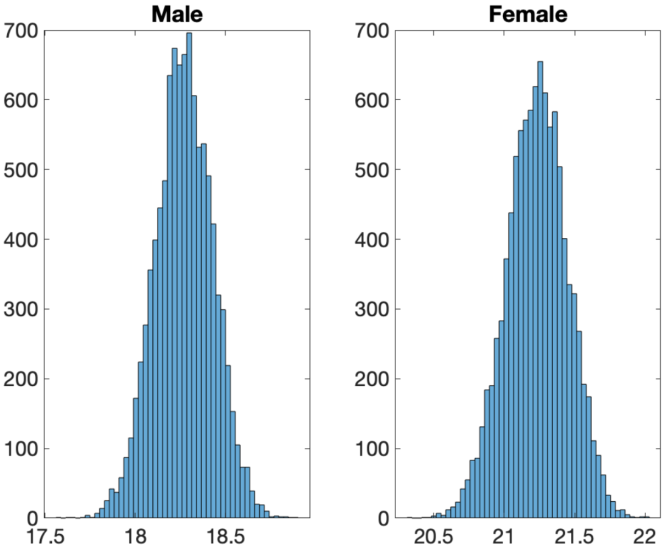
Implementation in Matlab

- To simulate a sample from a standard normal distribution, you could make use of the command

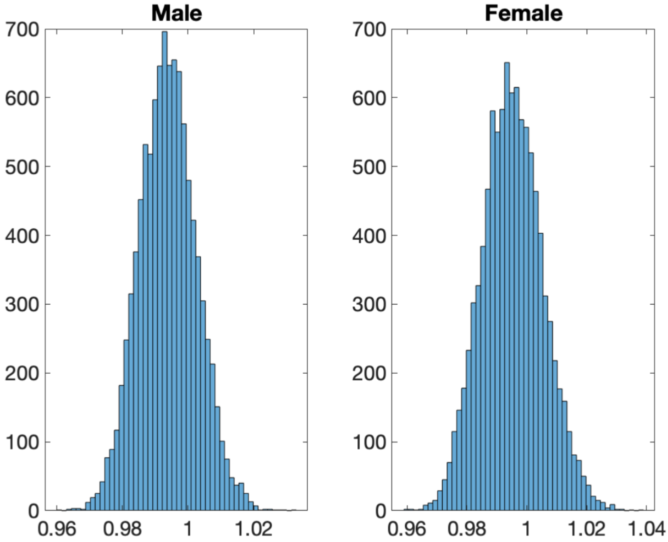
```
Z=randn(nrows,ncolumns).
```

- The matrix `Z` with dimension `nrows` $\times$ `ncolumns` then contains pseudo random numbers from a standard normal distribution.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, we can write $X = \mu + \sigma Z$, where $Z \sim \mathcal{N}(0, 1)$. This can be used to get a sample from a $\mathcal{N}(\mu, \sigma^2)$-distribution.

Histogram Annuity Factor



Histogram Funding Ratio



Interest Rate Risk: Assumptions

- We now adapt Assumption 3 on Slide 155 that returns are known in advance. We rather replace it by a term structure model for the evolution of interest rates.
- There are many ways to do this. As illustration, we shall make use of the Vasicek model, applied to term structure data provided by DNB (the Dutch Central Bank),

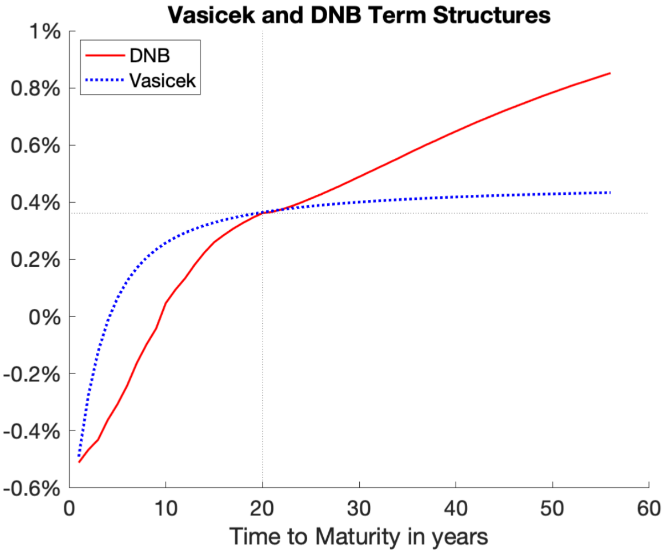
$$r_{t+1} = \mu + \theta r_t + \sigma \varepsilon_{t+1}, \qquad R_t(t+1) = r_t.$$

- Based on DNB-data from 2019, we find

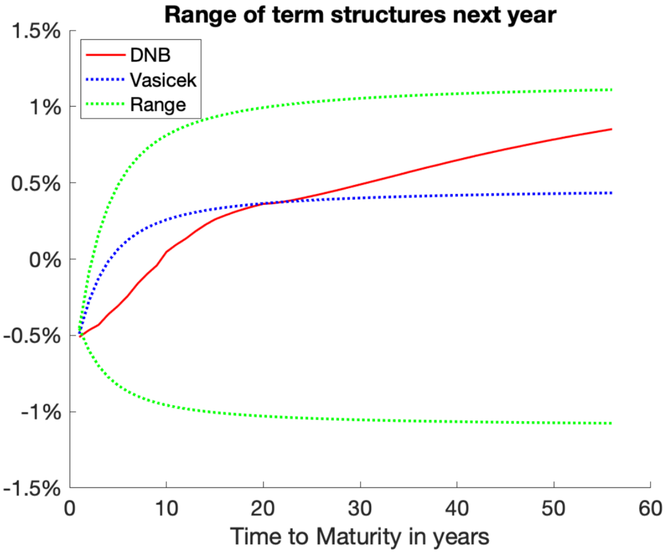
$$r_{t+1} = 0.0018 + 0.5522 r_t + 0.0026 \varepsilon_{t+1}.$$

- In the lecture Valuation and Risk Management (QFAS master), we are going to study various other interest rate risk models and discuss the pros and cons of these models.

Interest Rate Risk: Fit to the Data



Interest Rate Risk: Forecast



- Maintaining assumptions 1 and 2 on slide 155, we can now calculate the evolution of assets and liabilities.
- Assets (same calculations as before):

$$A_{t+1} = A_t(1+r_t) - \sum_{x\in\mathcal{X}} \underbrace{N_{x+1,t+1}}_{=N_{x,t}p_{x,t}}$$
$$= (1+r_t)\sum_{x\in\mathcal{X}} N_{x,t}a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}$$

- Liabilities (same calculations as before):

$$L_{t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x+1,t+1}a_{x+1,t+1}^{BE(t+1)} = \sum_{x\in\mathcal{X}} N_{x,t}p_{x,t}a_{x+1,t+1}^{BE(t+1)}$$

where

$$a_{x+1,t+1}^{BE(t+1)} = \sum_{\tau=1}^{\infty} {}_\tau p_{x+1,t+1}^{BE(t+1)} \frac{1}{\big(1+R_{t+1}(t+1+\tau)\big)^{\tau}}.$$

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- Consequently, the funding ratio becomes

$$FR_{t+1} = \frac{(1+r_t)\sum_{x\in\mathcal{X}} N_{x,t} a_{x,t}^{BE(t)} - \sum_{x\in\mathcal{X}} N_{x,t} p_{x,t}}{\sum_{x\in\mathcal{X}} N_{x,t} p_{x,t} a_{x+1,t+1}^{BE(t+1)}}.$$

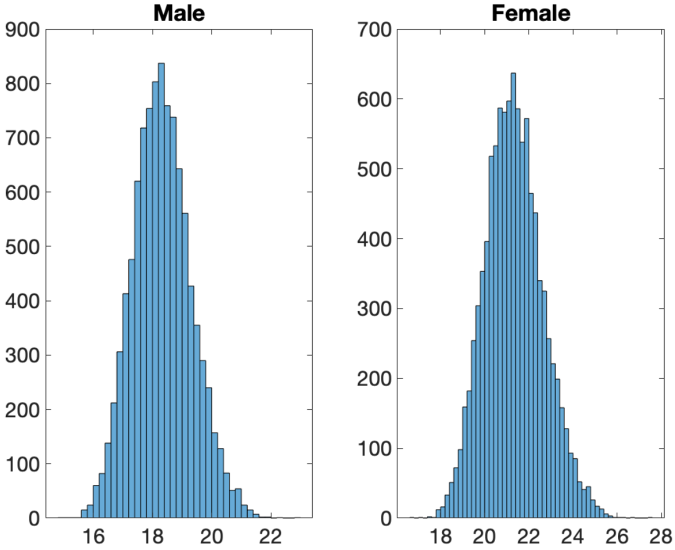
- To illustrate the effect of interest rate risk, we consider only one age group:

$$FR_{t+1} = \frac{(1+r_t) a_{x,t}^{BE(t)} - p_{x,t}}{p_{x,t} a_{x+1,t+1}^{BE(t+1)}}$$

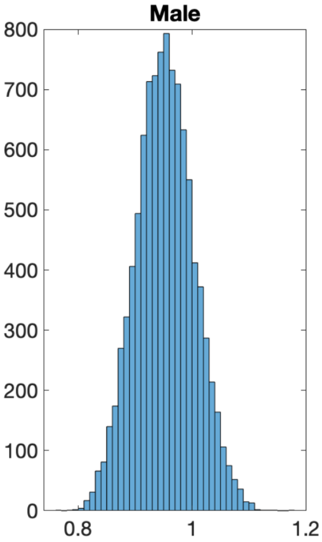
Illustration

- We illustrate interest rate risk using the following stylized example.
- We consider one age group ($x = 65$) at one specific year (namely, $t = 2019$).
- We present results for males and females separately.
- We generate 10,000 scenarios.
- We assume $r_{t+1} = 0.0018 + 0.5522 r_t + 0.0026 \varepsilon_{t+1}$ for all $\tau \geq 0$ and $r_{2019} = -0.51\%$.
- For the life table at time t, we use the Lee-Carter best estimate, based on the sample of Dutch males and females 1970-2019.
  - Males: $q_{65,2019}^{BE(2019)} = 1.07\%$, $p_{65,2019}^{BE(2019)} = 98.93\%$, $a_{65,2019}^{BE(2019)} = 18.95$.
  - Females: $q_{65,2019}^{BE(2019)} = 0.76\%$, $p_{65,2019}^{BE(2019)} = 99.24\%$, $a_{65,2019}^{BE(2019)} = 21.96$.

Histogram Annuity Factor



Histogram Funding Ratio

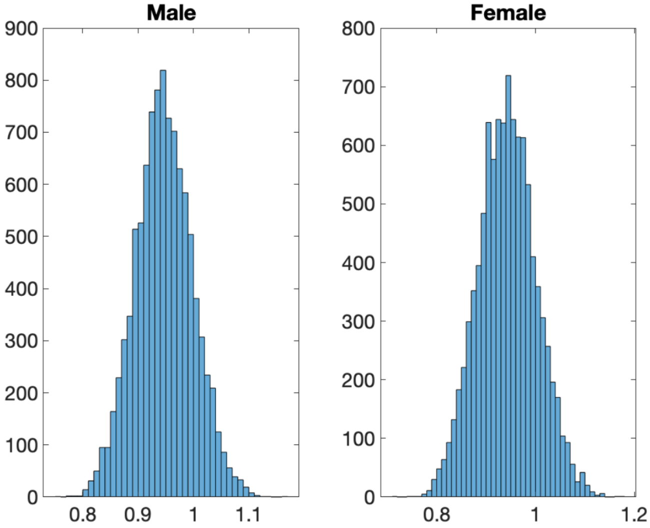


All Risks Combined

- We now switch on all types of risk as described in the previous sections.
- In addition, we add stock market risk and allow the pension fund to invest parts of its assets in risky stocks.
- We assume an annual volatility of $\sigma = 20\%$ and an expected annual rate of return of $\mu = 5\%$, where the return is normally distributed, i,e., $R_{t+1} \sim \mathcal{N}(\mu, \sigma^2)$ (Markowitz model).
- Assuming that the fund invests a fraction π of its assets in the stock market, the assets evolve according to

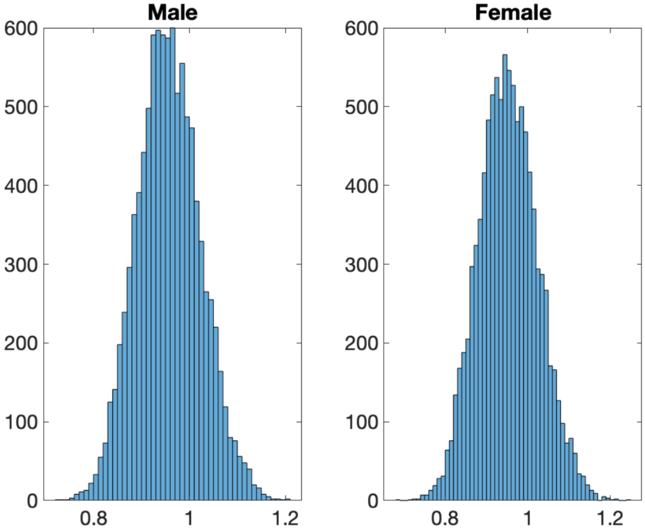
$$
\begin{aligned}
A_{t+1} &= A_t(1-\pi)(1+r_t) + A_t\pi(1+R_{t+1}) - \sum_{x\in\mathcal{X}} N_{x+1,t+1} \\
&= A_t[1 + r_t + \pi(R_{t+1} - r_t)] - \sum_{x\in\mathcal{X}} N_{x+1,t+1}.
\end{aligned}
$$

Histogram Funding Ratio



Investment Assets: $\pi = 0\%$ stock.

Histogram Funding Ratio



Investment Assets: $\pi = 20\%$ stock, with expected annual return $\mu = 5\%$ and annual volatility $\sigma = 20\%$.