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The Lee & Carter (1992) Model

- Lee and Carter (1992) directly model the evolution of the central death rate $m_{x,t}^{(g)}$ in a parsimonious way.
- Death rate evolves stochastically according to the dynamics

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)},$$

  where $\varepsilon_{x,t}^{(g)} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\varepsilon^{(g)}})$. $\alpha_x^{(g)}$ and $\beta_x^{(g)}$ are age-specific parameters.
- Dynamics of the latent factor, typically modeled as a random walk with drift:

$$\kappa_t^{(g)} = c^{(g)} + \kappa_{t-1}^{(g)} + \delta_t^{(g)},$$

  where $\delta_t^{(g)} \sim_{i.i.d.} \mathcal{N}(0, \sigma^2_{\delta^{(g)}})$.

Estimation

Given the estimates of $\alpha_x^{(g)}$, $\beta_x^{(g)}$, and $\kappa_t^{(g)}$, we can forecast the best estimates of the death rates, and hence the death probabilities:

$$\ln(\widehat{m}_{x,T+t_i}^{(g)}) = \widehat{\alpha}_x^{(g)} + \widehat{\beta}_x^{(g)} \widehat{\kappa}_{T+t_i}^{(g)}, \qquad i = 1, \ldots, N$$

Consequently,

$$\widehat{p}_{x,T+t_i}^{(g)} = \mathrm{e}^{-\widehat{m}_{x,T+t_i}^{(g)}}.$$

1. Original approach: Lee & Carter (1992):
   - Estimation of $\alpha_x^{(g)}$, $\beta_x^{(g)}$, $\kappa_t^{(g)}$ by *singular value decomposition* (SVD).
2. Iterative minimization of sum of squared errors:

$$\min_{\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}} \sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} - \beta_x^{(g)} \kappa_t^{(g)} \right]^2$$

Estimation: Original Approach

- The input to the model is a matrix of age-specific mortality rates (a life table of group g).
  - Let $\mathcal{X}$ be a set of size X of included ages, e.g., $\mathcal{X} = \{0, \dots, 90\}$, $X = 91$.
  - Let $\mathcal{T}$ be the set of size T of included periods, e.g., $\mathcal{T} = \{1970, \dots, 2021\}$, $T = 52$.
- **First step:** estimate $\alpha_x^{(g)}$ as the average over time of $\ln(m_{x,t}^{(g)})$, $x \in \mathcal{X}$, $t \in \mathcal{T}$:

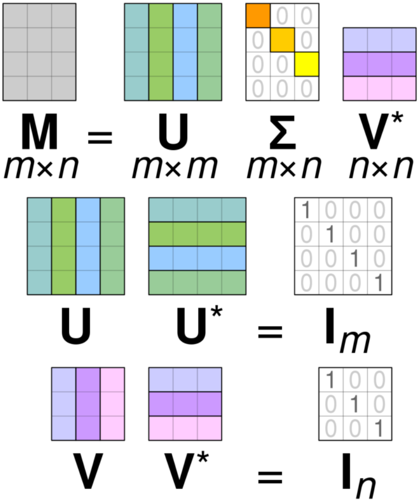
$$\widehat{\alpha}_x^{(g)} = \frac{1}{T} \sum_{t=1}^{T} \ln(m_{x,t}^{(g)})$$

- **Second step:** Calculate the matrix

$$M^{(g)} = \left(\ln(m_{x,t}^{(g)}) - \widehat{\alpha}_x^{(g)} \right)_{x \in \mathcal{X}, t \in \mathcal{T}} \in \mathbb{R}^{X \times T}$$

Estimation: First step

Estimation: Original Approach – SVD



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Excursion: Eckart-Yang-Mierski Theorem

Estimation: Original Approach – SVD

- **Third step:** Singular value decomposition can be performed numerically.
- Applying the singular value decomposition to the matrix

$$M^{(g)} = \left(\ln(m_{x,t}^{(g)}) - \widehat{\alpha}_x^{(g)}\right)_{x\in\mathcal{X},t\in\mathcal{T}} \in \mathbb{R}^{X\times T}$$

yields three matrices $U^{(g)} \in \mathbb{R}^{X\times X}$, $\Sigma^{(g)} \in \mathbb{R}^{X\times T}$, and $V^{(g)} \in \mathbb{R}^{T\times T}$ such that

$$M^{(g)} = U^{(g)}\Sigma^{(g)}V^{(g)\prime}.$$

- Unlike standard OLS estimation, this method is not plagued by the existence of multiple local minima. According to the Eckart-Young-Mirsky Theorem, the global minima can be directly obtained from the SVD decomposition.

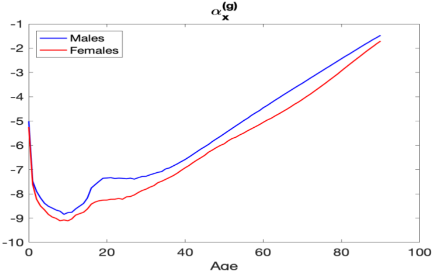
Estimation: Original Approach – SVD

- **Fourth step:** Calculate the estimates $\widehat{\beta}_x^{(g)}$, $\widehat{\kappa}_t^{(g)}$ from $U^{(g)} = (u_1^{(g)}, \dots, u_X^{(g)}) \in \mathbb{R}^{X \times X}$ and $V^{(g)} = (v_1^{(g)}, \dots, v_T^{(g)}) \in \mathbb{R}^{T \times T}$.
- Skipping all the technical details, one obtains under appropriate normalizations ($\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t \in \mathcal{T}} \kappa_t^{(g)} = 0$) the estimates

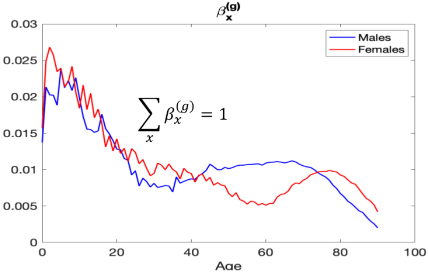
$$\widehat{\beta}_x^{(g)} = u_1^{(g)},$$
$$\widehat{\kappa}_t^{(g)} = \Sigma_{1,1}^{(g)} v_1^{(g)},$$

  where $\Sigma_{1,1}^{(g)}$ is the largest singular value.
- *Remark*: The original SVD approach is equivalent to LS estimation. We will discuss this in the tutorials.

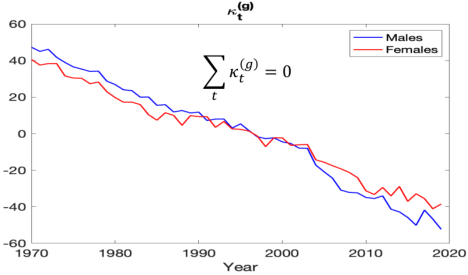
Estimates



$\alpha_x^{(g)}$
Males
Females
Age



$\beta_x^{(g)}$
Males
Females
$\sum_x \beta_x^{(g)} = 1$
Age



$\kappa_t^{(g)}$
Males
Females
$\sum_t \kappa_t^{(g)} = 0$
Year

- **Fifth step:** Simulate $\kappa^{(g)}_{T+t_i}$, $i = 1, \ldots, N$, i.e., for N additional years.
- One has to specify the dynamics of the time trend $\kappa^{(g)}_t$.
- Well established and typically a very good fit:

$$\kappa^{(g)}_t = c^{(g)} + \kappa^{(g)}_{t-1} + \delta^{(g)}_t$$

  Consequently, the time trend evolves like a random walk with drift

$$\Delta\kappa^{(g)}_t = c^{(g)} + \delta^{(g)}_t.$$

- Estimation:

$$\widehat{c}^{(g)} = \frac{1}{T-1}\sum_{t=2}^{T}\Delta\kappa^{(g)}_t = \frac{\kappa^{(g)}_T - \kappa^{(g)}_1}{T-1}.$$

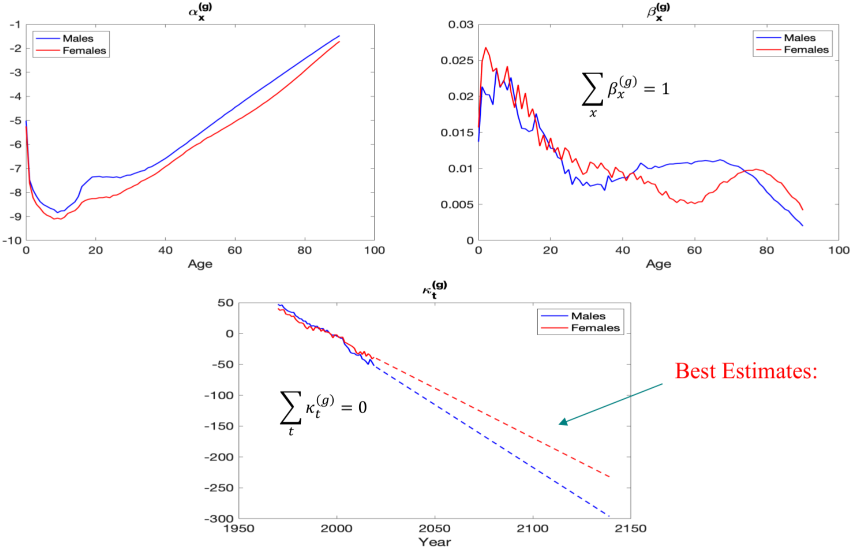
Forecasting

- **Sixth step:** Use the simulated data to predict the best estimates of the future death and survival probabilities and to estimate confidence intervals for these variables.
- Perform a Monte-Carlo simulation for $\kappa^{(g)}_{T+t_i}$ and $m^{(g)}_{x,T+t_i}$, i.e., simulate a large number of paths $\omega \in \Omega$, say $\mid \Omega \mid = 10,000$.
- Compute for each path $\omega \in \Omega$ the survival and death probabilities

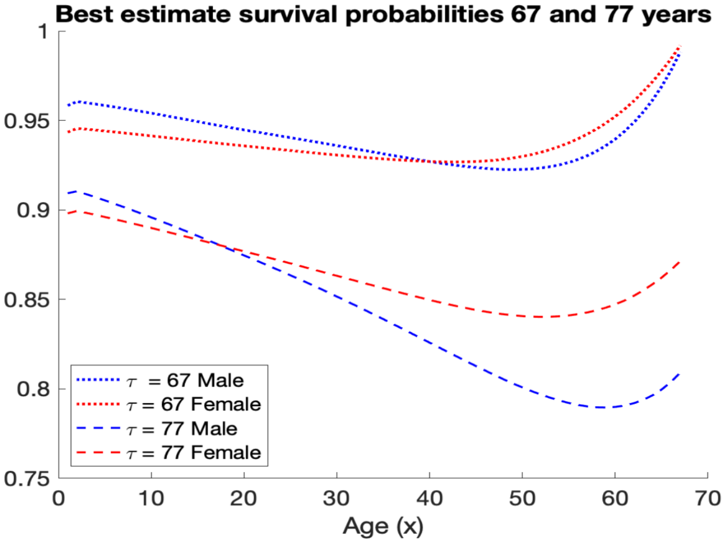
$$\widehat{p}^{(g)}_{x,T+t_i}(\omega) = \mathrm{e}^{-\widehat{m}^{(g)}_{x,T+t_i}(\omega)}, \qquad \widehat{q}^{(g)}_{x,T+t_i}(\omega) = 1 - \widehat{p}^{(g)}_{x,T+t_i}(\omega).$$

- Make a probability distribution (e.g., a histogram) for the forecasted probabilities.
- Derive the relevant moments from the resulting distribution such as mean, median, standard deviation, skewness, 5% and 95%-quantile,. . . .

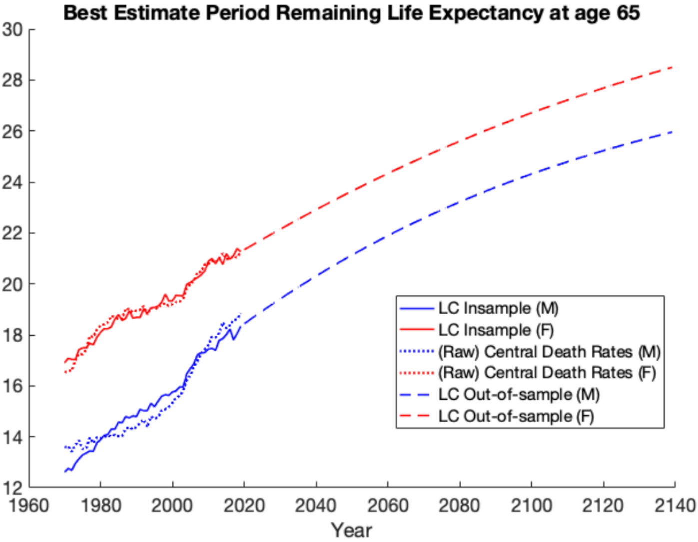
Best Estimate Forecast



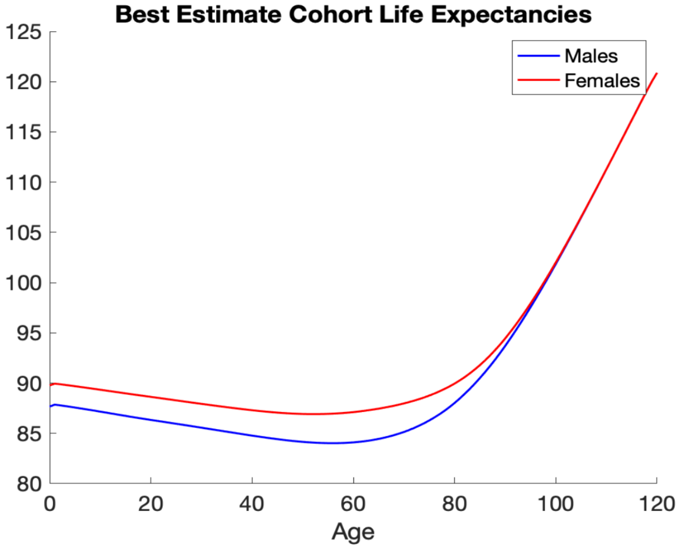
Best Estimate Survival Probabilities



Best Estimate Period Life Expectancy



Best Estimate Cohort Life Expectancy



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- We now consider future periods:

$$\Delta\kappa_t^{(g)} = c^{(g)} + \delta_t^{(g)}$$

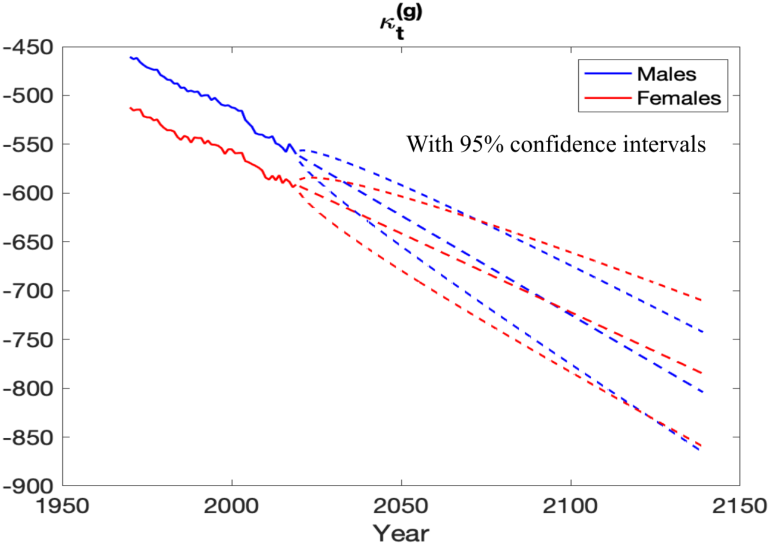
- These dynamics imply

$$\kappa_{T+t_i}^{(g)} = \underbrace{\kappa_T^{(g)} + t_i \cdot c^{(g)}}_{\text{Best Estimate}} \quad + \quad \underbrace{\sum_{j=1}^{i} \delta_{T+t_j}^{(g)}}_{\text{Forecast Error}} \quad .$$

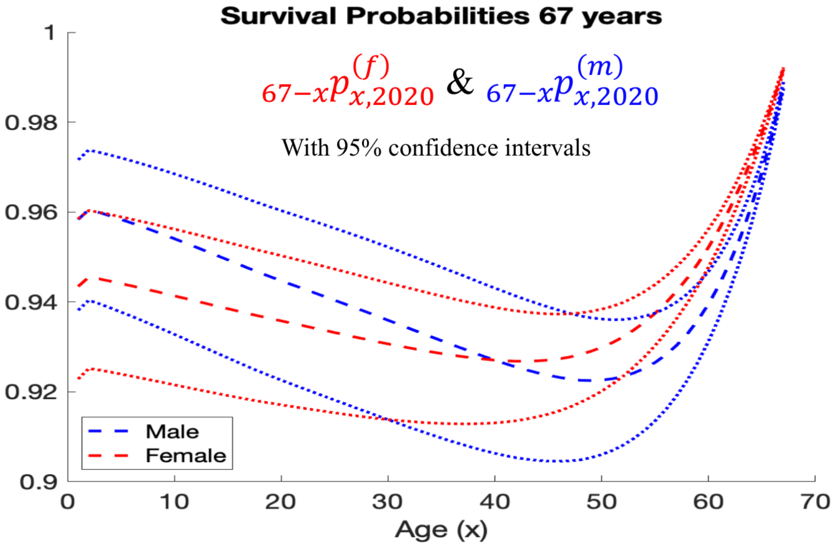
- Because $\delta_t^{(g)} \sim_{i.i.d.} \mathcal{N}(0, \sigma_{\delta(g)}^2)$, the distribution of the trend component is

$$\kappa_{T+t_i}^{(g)} \sim \mathcal{N}\Big(\kappa_T^{(g)} + t_i \cdot c^{(g)}, t_i \cdot \sigma_{\delta(g)}^2\Big).$$

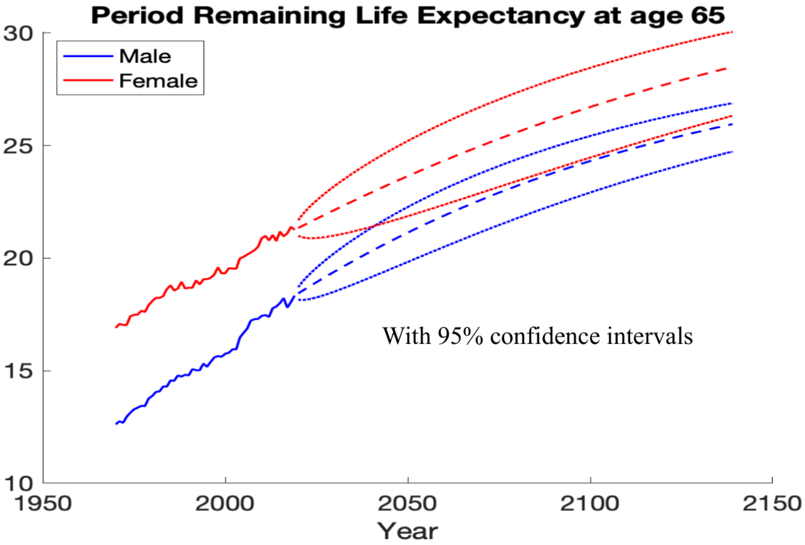
Macro Longevity Risk – Forecast



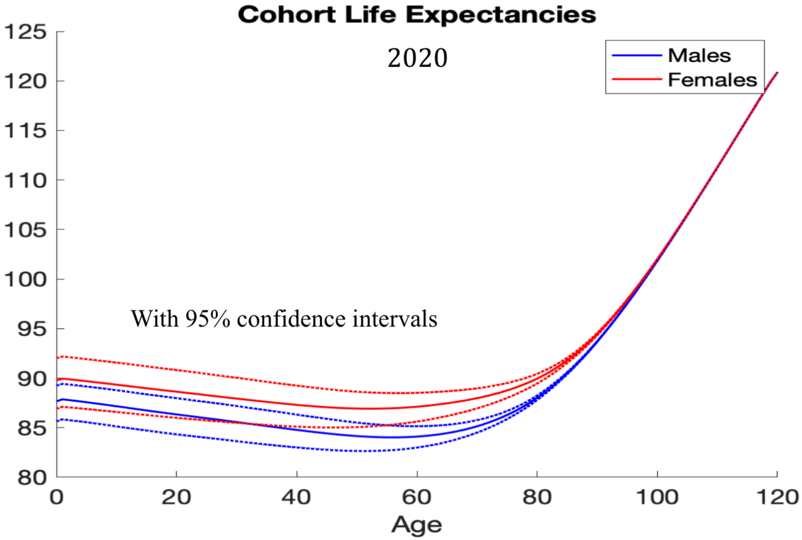
Macro Longevity Risk – Survival Probabilities



Macro Longevity Risk – Period Life Expectancy



Macro Longevity Risk – Cohort Life Expectancy



Ordinary Least Squares

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- Assume the Lee-Carter normalization ($\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t\in\mathcal{T}} \kappa_t^{(g)} = 0$).
- In Problem 9, it is to show that the estimation of $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$ by the Singular Value Decomposition (SVD) is the same as minimizing the sum of squared errors

$$\min_{\alpha_x^{(g)},\beta_x^{(g)},\kappa_t^{(g)}} \sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \alpha_x^{(g)} - \beta_x^{(g)}\kappa_t^{(g)} \right]^2$$

  w.r.t $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$.
- This minimization is typically done iteratively:
  - minimize w.r.t. $\alpha_x^{(g)}$ (all x), keeping all $\beta_x^{(g)}, \kappa_t^{(g)}$ fixed,
  - minimize w.r.t. $\beta_x^{(g)}$ (all x), keeping all $\alpha_x^{(g)}, \kappa_t^{(g)}$ fixed,
  - minimize w.r.t. $\kappa_t^{(g)}$ (all t), keeping all $\alpha_x^{(g)}, \beta_x^{(g)}$ fixed,
  - keep iterating until convergence.
- We will now dive deeper into this issue.

- If $\widehat{\alpha}_x^{(g)}, \widehat{\beta}_x^{(g)}, \widehat{\kappa}_t^{(g)}$ minimize the sum of squared errors

$$\sum_{x,t} \left[\ln(m_{x,t}^{(g)}) - \widehat{\alpha}_x^{(g)} - \widehat{\beta}_x^{(g)} \widehat{\kappa}_t^{(g)} \right]^2,$$

then also $\widetilde{\alpha}_x^{(g)} = \widehat{\alpha}_x^{(g)} + c_1 \widehat{\beta}_x^{(g)}$, $\widetilde{\beta}_x^{(g)} = c_2 \widehat{\beta}_x^{(g)}$, $\widetilde{\kappa}_t^{(g)} = \frac{\widehat{\kappa}_t^{(g)}}{c_2} - \frac{c_1}{c_2}$.

- So, we need two normalizations. Standard normalizations: $\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t \in \mathcal{T}} \kappa_t^{(g)} = 0$.
- In the sequel we shall compare the standard normalization to the normalization of Liu et al. (2019a & b), i.e., $\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{x \in \mathcal{X}} \alpha_x^{(g)} = 0$ (see Problem 11).

Interpretation of Normalizations

Standard Normalization (Lee-Carter)

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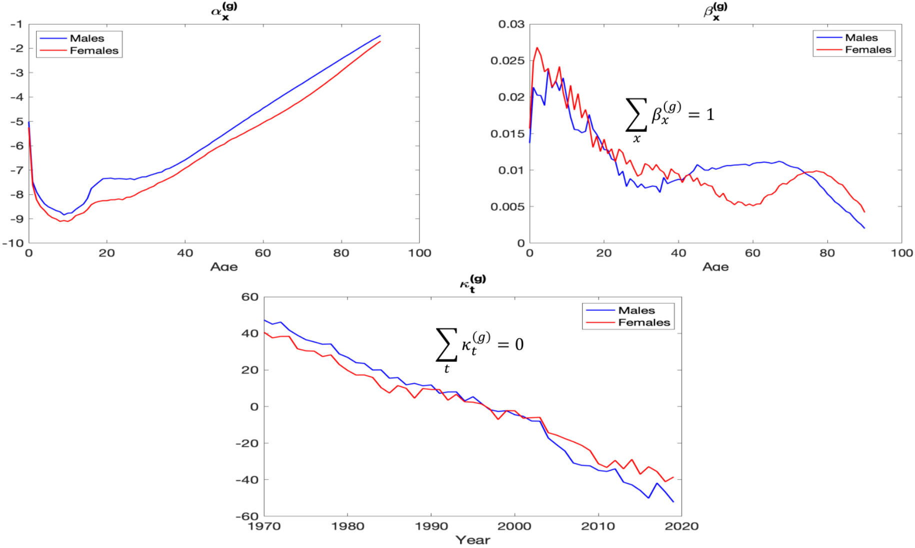
- Choose starting values for $\beta_x^{(g)}, \kappa_t^{(g)}$ (with $\alpha_x^{(g)} = \frac{1}{T}\sum_{t=1}^{T}\ln(m_{x,t}^{(g)})$ and under the standard normalization $\sum_{x\in\mathcal{X}}\beta_x^{(g)} = 1$ and $\sum_{t\in\mathcal{T}}\kappa_t^{(g)} = 0$).
- Take average over x:

- Take time differences and then average over t:

Estimates (Lee and Carter 1992)



Alternative Normalization (Liu et al. 2019)

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- Liu et al. (2019a,b) propose an alternative normalization which makes estimation of the Lee-Carter model possible using linear regressions.
- Choose starting values for $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$ (under the Liu normalization $\sum_{x \in \mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{x \in \mathcal{X}} \alpha_x^{(g)} = 0$).
- Take sum over x:
- Run regressions for each x:

Alternative Normalization (Liu et al. 2019)

- Start from the Lee-Carter model $\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}$ combined with the normalization $\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{x\in\mathcal{X}} \alpha_x^{(g)} = 0$.
- Define

$$\begin{aligned} Z_t^{(g)} &= \sum_{x\in\mathcal{X}} \ln(m_{x,t}^{(g)}) = \sum_{x\in\mathcal{X}} \alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)} + \varepsilon_{x,t}^{(g)} \\ &= \kappa_t^{(g)} + \underbrace{\sum_{x\in\mathcal{X}} \varepsilon_{x,t}^{(g)}}_{=e_t^{(g)}} . \end{aligned}$$

- Assume now a random walk with drift for $\kappa_t^{(g)}$:

$$\begin{aligned} & \kappa_t^{(g)} = c^{(g)} + \kappa_{t-1}^{(g)} + \delta_t^{(g)} \\ \Longleftrightarrow \quad & Z_t^{(g)} = c^{(g)} + Z_{t-1}^{(g)} + \delta_t^{(g)} + e_t^{(g)} - e_{t-1}^{(g)}. \end{aligned}$$

Alternative Normalization (Liu et al. 2019)

- Next, we substitute

$$Z_t^{(g)} = c^{(g)} + Z_{t-1}^{(g)} + \left(\delta_t^{(g)} + e_t^{(g)} - e_{t-1}^{(g)}\right)$$

into

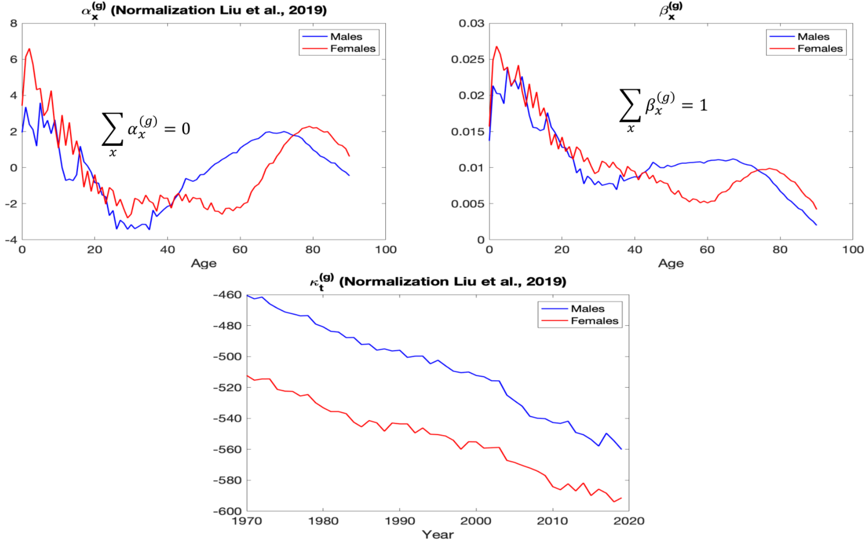
$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} \kappa_t^{(g)} + \varepsilon_{x,t}^{(g)}.$$

- Thus, we arrive at

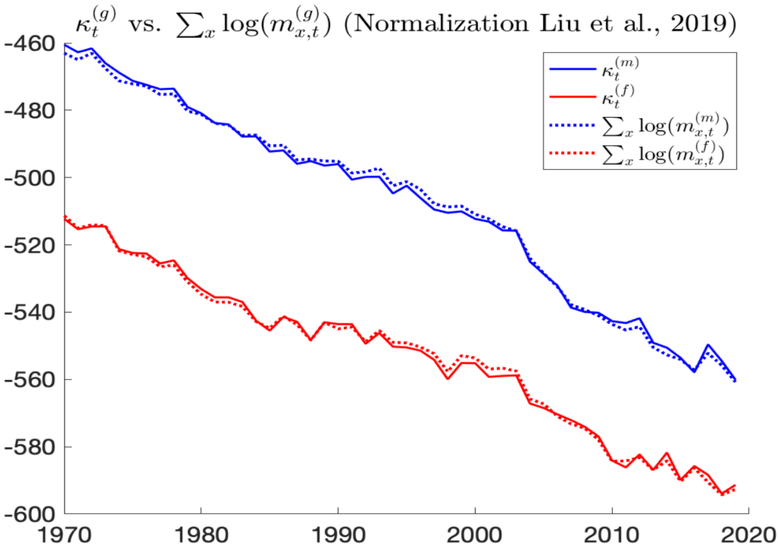
$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)} Z_t^{(g)} + \left(\varepsilon_{x,t}^{(g)} - \beta_x^{(g)} e_t^{(g)}\right),$$
$$Z_t^{(g)} = c^{(g)} + Z_{t-1}^{(g)} + \left(\delta_t^{(g)} + e_t^{(g)} - e_{t-1}^{(g)}\right).$$

- The parameters in these equations can be estimated in a single step, applying standard linear regression techniques (possibly using instrumental variables).

Estimates (Liu et al. 2019)



A Comparison



Motivation

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- Lee-Carter is not the end of the story. It is rather the benchmark model that can be extended and modified into many dimensions.
- Modeling is always a trade-off between:
  - Tractability
  - Parsimony
  - Accuracy
- Unfortunately, there is no unified model that works best in all countries and in all populations. Instead, some of the variants work better in some countries, others work better in other countries.
- We will now briefly discuss some of the extensions and alternatives:
  - Trend correction
  - Alternative estimation approaches
  - Jump-off bias correction
  - Alternative models
  - Multi-population models

Trend Correction

- Lee and Carter proposed to correct the estimated trend to fit the actual number of deaths, i.e., replace $\widehat{\kappa}_t^{(g)}$ by $\widetilde{\kappa}_t^{(g)}$ where $\widetilde{\kappa}_t^{(g)}$ is chosen such that

$$\sum_{x\in\mathcal{X}} D_{x,t}^{(g)} = \sum_{x\in\mathcal{X}} E_{x,t}^{(g)} \exp\left(\widehat{\alpha}_x^{(g)} + \widehat{\beta}_x^{(g)}\widetilde{\kappa}_t^{(g)}\right).$$

- Alternative estimation methods (see next slide for illustration) incorporate this correction step.

Recall: Notation

- Instead of SVD or OLS estimation, one can also apply a maximum likelihood approach to estimate the parameters $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$.
  - Avoids homoscedasticity of the errors occurring in OLS/SVD estimation.
  - Notice that the logarithm of the observed force of mortality is much more variable at older ages than at younger ages.
- Brouhns et al. (2002) model the number of deaths $D_{x,t}^{(g)}$ as Poisson-distributed random variables.
- Remember

$$\widehat{\mu}_{x,t}^{(g)} = m_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}.$$

- A reasonable model for $D_{x,t}^{(g)}$ would thus be

$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}(\mu_{x,t}^{(g)} E_{x,t}^{(g)}).$$

- Consequently,

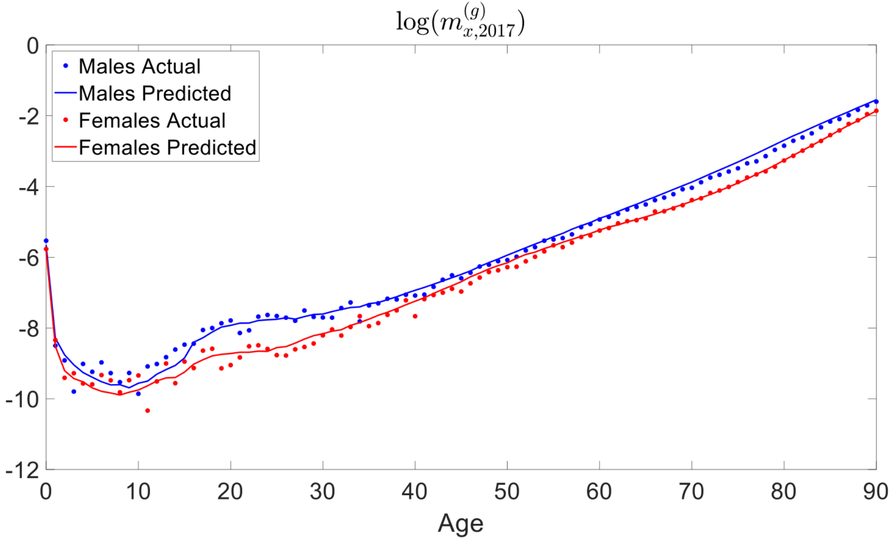
$$D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}\big(\mathrm{e}^{\alpha_x^{(g)}+\beta_x^{(g)}\kappa_t^{(g)}} E_{x,t}^{(g)}\big).$$

- Set up the log-likelihood function over all observations (x,t) and maximize it with respect to all $\alpha_x^{(g)}, \beta_x^{(g)}, \kappa_t^{(g)}$:

$$\ell(\alpha,\beta,\kappa) = \sum_{(x,t)\in\mathcal{X}\times\mathcal{T}} \Big[D_{x,t}^{(g)} \big(\alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)}\big) - E_{x,t}^{(g)} \mathrm{e}^{\alpha_x^{(g)}+\beta_x^{(g)}\kappa_t^{(g)}} \Big].$$

- This maximization has again to be done iteratively. Normalizations such as $\sum_{x\in\mathcal{X}} \beta_x^{(g)} = 1$ and $\sum_{t\in\mathcal{T}} \kappa_t^{(g)} = 0$ are also needed again.
- This method is applied in the AG2022 model.

Jump-off Bias



Jump-off Bias Correction

- Lee-Carter model: $\widehat{m}^{(g)}_{x,T+t_i} = \exp\left(\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i}\right)$, $i = 1, \ldots, N$.
- *Problem*: possibility of jump-off bias, i.e., $\widehat{m}^{(g)}_{x,T} \neq m^{(g)}_{x,T}$, where the latter is observable.
- To avoid this *jump-off bias* shift $\ln \widehat{m}^{(g)}_{x,T+t_i} = \widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i}$ by

$$\ln m^{(g)}_{x,T} - (\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_T)$$

  to get an alternative forecast (correcting for jump-off bias):

$$\begin{aligned} \ln \widetilde{m}^{(g)}_{x,T+t_i} &= \widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i} + \ln m^{(g)}_{x,T} - (\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_T) \\ &= \ln m^{(g)}_{x,T} + \widehat{\beta}^{(g)}_x \left(\widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T\right). \end{aligned}$$

Jump-off Bias Correction

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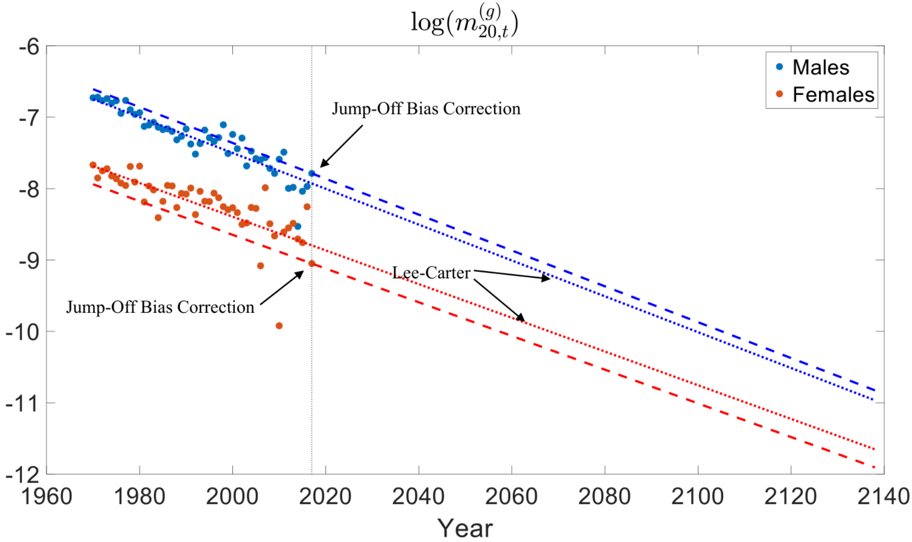

- Going from the forecast $\widehat{m}^{(g)}_{x,T+t_i} = \exp\left(\widehat{\alpha}^{(g)}_x + \widehat{\beta}^{(g)}_x \widehat{\kappa}^{(g)}_{T+t_i}\right)$, $i = 1, \ldots, N$ to

$$\ln \widetilde{m}^{(g)}_{x,T+t_i} = \ln m^{(g)}_{x,T} + \widehat{\beta}^{(g)}_x \left(\widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T\right)$$

  means replacing
  - $\widehat{\alpha}^{(g)}_x = \frac{1}{T}\sum_{t\in\mathcal{T}} \ln m^{(g)}_{x,t}$ by $\widetilde{\alpha}^{(g)}_x = \ln m^{(g)}_{x,T}$,
  - $\widehat{\kappa}^{(g)}_{T+t_i}$ by $\widetilde{\kappa}^{(g)}_{T+t_i} = \widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T$
- This way of avoiding the jump-off bias corresponds to the use of *reduction factors*

$$\widetilde{m}^{(g)}_{x,T+t_i} = m^{(g)}_{x,T} \cdot \mathrm{RF}^{(g)}_{x,t_i}$$

  with $\mathrm{RF}^{(g)}_{x,t_i} = \exp\left(\widehat{\beta}^{(g)}_x [\widehat{\kappa}^{(g)}_{T+t_i} - \widehat{\kappa}^{(g)}_T]\right)$.

- **Extensions of the Lee-Carter model**
  - Extra factor(s):

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_{1,x}^{(g)}\kappa_{1,t}^{(g)} + \beta_{2,x}^{(g)}\kappa_{2,t}^{(g)} + \ldots + \varepsilon_{x,t}^{(g)}$$

  - Cohort effect:

$$\ln(m_{x,t}^{(g)}) = \alpha_x^{(g)} + \beta_x^{(g)}\kappa_t^{(g)} + \gamma_x^{(g)}\iota_{t-x}^{(g)} + \varepsilon_{x,t}^{(g)}$$

- **Modeling alternative indicators**
  - Cairns-Blake-Dowd (CBD-)models, see Exercise 15:

$$\ln\left(\frac{q_{x,t}^{(g)}}{p_{x,t}^{(g)}}\right)$$

  - Oeppen (2008):

$$\ln\left(\frac{D_{x,t}^{(g)}}{h_t^{(g)}}\right), \qquad \text{with} \qquad h_t^{(g)} = \sqrt[X]{\prod_{x=1}^{X} D_{x,t}^{(g)}}$$

Interpretation of the Time Trend

Model Performance in Different Countries

Average root mean square error ($\overline{RMSE}$) over forecast horizon h of the forecast life expectancy at birth for the period 1990 to 2014, with the best $RMSE$ value per country in bold and preferred set of models (SP) for 18 countries, females and males

Females							Males						
Country	**M**	**Q**	**D**	**L**	**E**	**SP**	**Country**	**M**	**Q**	**D**	**L**	**E**	**SP**
DEU–E	1.26	1.25	1.17	1.13	**1.01**	M, Q, D, L, E	DEU–E	2.35	2.35	2.22	2.11	**1.93**	E
DNK	1.17	1.15	**0.76**	0.91	0.96	D	DNK	2.16	2.16	2.08	2.03	**1.88**	E
IRL	1.08	1.07	0.87	0.86	**0.61**	E	IRL	2.13	2.12	2.09	1.90	**1.67**	E
PRT	0.77	0.75	0.45	**0.35**	1.10	L	NLD	1.75	1.75	1.74	1.68	**1.47**	E
JPN	**0.67**	0.72	1.88	1.59	1.57	M	NOR	1.71	1.71	1.66	1.62	**1.43**	E
UK	0.65	0.64	**0.38**	0.47	0.40	D	PRT	1.50	1.49	1.11	0.78	**0.44**	E
NOR	0.50	0.48	**0.24**	0.28	0.26	D	ITA	1.28	1.27	0.96	0.92	**0.52**	E
NLD	0.50	0.49	0.56	0.51	**0.49**	M, Q, D, L, E	ESP	1.25	1.23	0.81	0.78	**0.52**	E
AUT	0.47	0.46	0.36	**0.33**	0.33	M, Q, D, L, E	UK	1.23	1.22	0.93	0.98	**0.88**	D, E
USA	**0.43**	0.44	0.66	0.57	0.60	M, Q	CHE	1.20	1.19	0.79	0.89	**0.78**	D, E
ITA	0.31	**0.30**	0.52	0.41	0.76	Q	AUT	1.11	1.11	0.85	0.83	**0.62**	E
AUS	0.27	**0.26**	0.38	0.30	0.40	M, Q, L	SWE	1.08	1.08	0.93	0.95	**0.81**	E
ESP	0.24	**0.23**	0.33	0.34	0.87	Q	FRA	1.03	1.02	**0.53**	0.66	0.65	D
FRA	**0.23**	0.23	0.69	0.52	0.59	M	DEU–W	0.92	0.91	0.67	0.66	**0.42**	E
FIN	**0.22**	0.22	0.84	0.54	0.46	M, Q	AUS	0.89	0.86	0.52	0.55	**0.48**	E
DEU–W	**0.21**	0.21	0.64	0.49	0.65	M	FIN	0.87	0.86	**0.42**	0.56	0.47	D
CHE	**0.21**	0.22	0.82	0.59	0.58	M	JPN	**0.59**	0.62	1.59	1.23	1.25	M
SWE	**0.17**	0.18	0.54	0.38	0.38	M	USA	0.49	0.48	0.27	0.28	**0.24**	E
Mean	0.52	**0.52**	0.67	0.59	0.67		Mean	1.31	1.30	1.12	1.08	**0.92**	

M.-P. Bergeron-Boucher, S. Kjærgaard, J. Oeppen, J.W. Vaupel (2019)

- **Li and Lee (2005)**
  - Different populations do not live in isolation. Instead, there is a lot of interaction.
  - Therefore, it seems implausible that the mortalities of similar populations will diverge in the long run.
  - Similar populations have a common (non-stationary) time trend, while the difference between each population's time trend and the common time trend is likely stationary.
  - Traditional estimation to be done in multiple steps.

- **Application: AG2022-Model**
  - Similar populations determined based on GDP per capita.