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6. Model Risk: A Very Brief Introduction

Remaining Lifetime

- Let $T_{x,t}^{(g)}$ denote the (random) remaining lifetime of somebody of age x, group g, at time t with probability distribution function

$$F_{x,t}^{(g)}(\tau) = \mathbb{P}(T_{x,t}^{(g)} \leq \tau).$$

- Assuming that the remaining lifetime has a probability density function $f_{x,t}^{(g)}$ such that

$$\mathbb{P}(\tau < T_{x,t}^{(g)} < \tau + \mathrm{d}\tau) = f_{x,t}^{(g)}(\tau)\mathrm{d}\tau.$$

- τ-years survival probability of an individual of age x:

$${}_\tau p_{x,t}^{(g)} = 1 - F_{x,t}^{(g)}(\tau)$$

Remaining Lifetime

Force of Mortality

- The force of mortality (or hazard rate of death) is defined as

$$h_{x,t}^{(g)}(\tau) = \mu_{x+\tau,t+\tau}^{(g)} = -\frac{\partial}{\partial\tau}\ln({}_{\tau}p_{x,t}^{(g)})$$

representing the instantaneous rate of mortality at a certain age measured on an annualized basis.

- Formally,

$$\mu_{x+\tau,t+\tau}^{(g)} = \frac{f_{x,t}^{(g)}(\tau)}{1 - F_{x,t}^{(g)}(\tau)}.$$

- The instantaneous survival probability can be rewritten in terms of the hazard rate

$$\mathbb{P}(\tau < T_{x,t}^{(g)} < \tau + \mathrm{d}\tau) = f_{x,t}^{(g)}(\tau)\mathrm{d}\tau = \mu_{x+\tau,t+\tau}^{(g)} \cdot {}_{\tau}p_{x,t}^{(g)}\mathrm{d}\tau$$

Force of Mortality: Intuition

- Integrating the force of mortality $\mu^{(g)}_{x+\tau,t+\tau} = -\frac{\partial}{\partial\tau}\ln({}_{\tau}p^{(g)}_{x,t})$ yields

$$\int_0^s \mu^{(g)}_{x+\tau,t+\tau}\mathrm{d}\tau = -\int_0^s \frac{\partial}{\partial\tau}\ln({}_{\tau}p^{(g)}_{x,t})\mathrm{d}\tau = -\ln({}_{s}p^{(g)}_{x,t}).$$

- Consequently,

$${}_{s}p^{(g)}_{x,t} = \exp\Big(-\int_0^s \mu^{(g)}_{x+\tau,t+\tau}\mathrm{d}\tau\Big).$$

- Assumption: $\mu^{(g)}_{x+\tau,t+\tau} = \mu^{(g)}_{x,t}$ if $0 \leq \tau < 1$.
- Under this assumption, the survival probability is

$${}_{s}p^{(g)}_{x,t} = \exp\big(-s\cdot\mu^{(g)}_{x,t}\big).$$

- In particular, the one-year survival probability can be rewritten as

$$p^{(g)}_{x,t} = {}_{1}p^{(g)}_{x,t} = \exp\big(-\mu^{(g)}_{x,t}\big).$$

Decomposition

- Let $\tau = \tau_1 + \tau_2$. Then, the τ-years survival probability is given by

$$ {}_{\tau}p_{x,t}^{(g)} = {}_{\tau_1}p_{x,t}^{(g)} \cdot {}_{\tau_2}p_{x+\tau_1,t+\tau_1}^{(g)}. $$

- Proof:

Decomposition

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- In general, the τ-years survival probability can be decomposed into

$$_{\tau}p_{x,t}^{(g)} = \prod_{k=0}^{\tau-1} p_{x+k,t+k}^{(g)}, \qquad p_{x,t}^{(g)} = 1 - q_{x,t}^{(g)}.$$

- Here, the one-year survival and death probabilities can be rewritten as

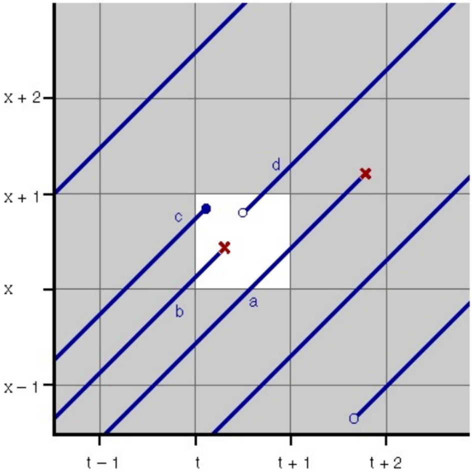
$$p_{x,t}^{(g)} = \exp(-\mu_{x,t}^{(g)}),$$
$$q_{x,t}^{(g)} = 1 - \exp(-\mu_{x,t}^{(g)}).$$

- **Moral**: Modeling the force of mortality is sufficient to model the survival and death probabilities.

  $\rightarrow$ How to estimate the historical force of mortality?

  $\rightarrow$ How to model the evolution of the force of mortality?

Example: Lexis Diagram

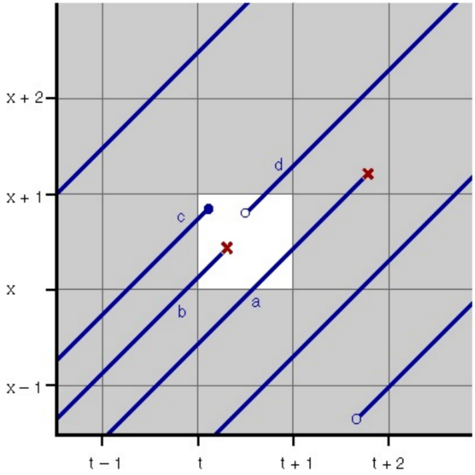


- Let $n_{x,t}^{(g)}$ lives contribute to the observations in the white square (Lexis diagram).
- Assume life $i \in N_{x,t}^{(g)} = \{1, \ldots, n_{x,t}^{(g)}\}$ is observed between the ages of $x + t_i$ and $x + s_i$ (with $0 \leq t_i < s_i \leq 1$).
- The *exposure* $E_{x,t}^{(g)}$ is defined as

$$E_{x,t}^{(g)} = \sum_{i=1}^{n_{x,t}^{(g)}} (s_i - t_i).$$

- $D_{x,t}^{(g)} \in \mathbb{N}$ denotes the *number of deaths* observed in the white square.
- $I_{x,t}^{(g)} \subseteq N_{x,t}^{(g)}$ is the subset of observations i terminated by death.

Example: Lexis Diagram



Likelihood Function

- Assume that the $n_{x,t}^{(g)}$ lives are independent, i.e., their remaining lifetimes $T_{x,t}^{(g)}$ are stochastically independent.
- The likelihood of the $n_{x,t}^{(g)}$ observations is

$$L_{x,t}^{(g)} = \prod_{i \in I_{x,t}^{(g)}} f_{x+t_i,t}^{(g)}(s_i - t_i) \prod_{i \notin I_{x,t}^{(g)}} \Big(1 - F_{x+t_i,t}^{(g)}(s_i - t_i)\Big)$$

$$= \prod_{i \in I_{x,t}^{(g)}} \mu_{x+s_i,t+s_i-t_i}^{(g)} \cdot {}_{s_i-t_i}p_{x+t_i,t}^{(g)} \prod_{i \notin I_{x,t}^{(g)}} {}_{s_i-t_i}p_{x+t_i,t}^{(g)}.$$

- Using our standing assumption $\mu_{x+s_i,t+s_j}^{(g)} = \mu_{x,t}^{(g)}$, $s_i, s_j \in [0,1]$:

$${}_{s_i-t_i}p_{x+t_i,t}^{(g)} = \exp\Big(-(s_i - t_i)\mu_{x+s_i,t+s_i-t_i}^{(g)}\Big) = \exp\Big(-(s_i - t_i)\mu_{x,t}^{(g)}\Big)$$

Likelihood Function

- Consequently, the likelihood can be expressed in terms of $\mu_{x,t}^{(g)}$:

$$\begin{aligned} L_{x,t}^{(g)} &= \prod_{i \in I_{x,t}^{(g)}} \mu_{x,t}^{(g)} \exp\Big(-(s_i - t_i)\mu_{x,t}^{(g)} \Big) \prod_{i \notin I_{x,t}^{(g)}} \exp\Big(-(s_i - t_i)\mu_{x,t}^{(g)} \Big) \\ &= \prod_{i \in I_{x,t}^{(g)}} \mu_{x,t}^{(g)} \prod_{i \in N_{x,t}^{(g)}} \exp\Big(-(s_i - t_i)\mu_{x,t}^{(g)} \Big) \\ &= \big[\mu_{x,t}^{(g)}\big]^{D_{x,t}^{(g)}} \exp\Big(-\sum_{i \in N_{x,t}^{(g)}} (s_i - t_i)\mu_{x,t}^{(g)} \Big). \end{aligned}$$

- Recall: exposure is defined as $E_{x,t}^{(g)} = \sum_{i=1}^{n_{x,t}^{(g)}} (s_i - t_i)$, i.e.,

$$L_{x,t}^{(g)} = \big[\mu_{x,t}^{(g)}\big]^{D_{x,t}^{(g)}} \exp\Big(-E_{x,t}^{(g)} \mu_{x,t}^{(g)} \Big).$$

Maximum Likelihood Estimate

- Maximizing the likelihood function

$$L_{x,t}^{(g)} = \left[\mu_{x,t}^{(g)}\right]^{D_{x,t}^{(g)}} \exp\left(- E_{x,t}^{(g)} \mu_{x,t}^{(g)} \right)$$

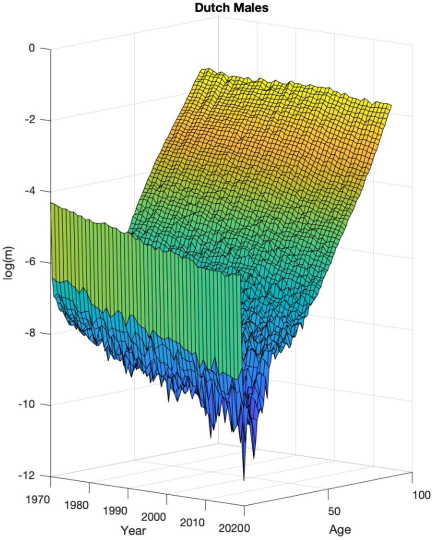
w.r.t. $\mu_{x,t}^{(g)}$ results in the maximum likelihood estimate

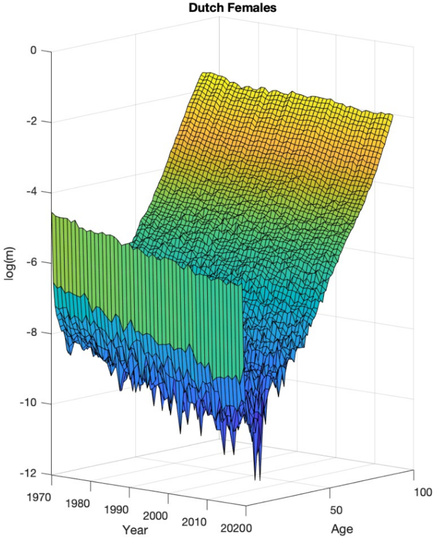
$$\widehat{\mu}_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}.$$

- The ratio $\frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}$ is called the *raw central death rate* and usually denoted by $m_{x,t}^{(g)}$.

Proof: Maximum Likelihood Estimate

Central Death Rates





1. For past and present periods, we estimate

$$\widehat{p}_{x,t}^{(g)} = \exp(-m_{x,t}^{(g)}),$$
$$\widehat{q}_{x,t}^{(g)} = 1 - \exp(-m_{x,t}^{(g)})$$

with $m_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}$.

2. For future periods, $D_{x,t+\dots}$ and $E_{x,t+\dots}$ are unobserved, and we need *models* to predict the future probabilities

$$p_{x,t}^{(g)} = \exp(-\mu_{x,t}^{(g)}),$$
$$q_{x,t}^{(g)} = 1 - \exp(-\mu_{x,t}^{(g)}).$$

- Lee & Carter (1992) and others: directly model $m_{x,t}^{(g)}$.
- AG2022 and others: model $D_{x,t}^{(g)} \mid E_{x,t}^{(g)} \sim \mathcal{P}(\mu_{x,t}^{(g)} E_{x,t}^{(g)})$.

Life Table of Group g

$$
\begin{array}{c}
\rightarrow \text{ time} \\
\downarrow \\
\text{age}
\end{array}
\begin{pmatrix}
q^{(g)}_{0,t_0} & q^{(g)}_{0,t_0+1} & \cdots & q^{(g)}_{0,t_0+T} & \vdots & \cdots \\
q^{(g)}_{1,t_0} & q^{(g)}_{1,t_0+1} & \cdots & q^{(g)}_{1,t_0+T} & \vdots & \cdots \\
q^{(g)}_{2,t_0} & q^{(g)}_{2,t_0+1} & \cdots & q^{(g)}_{2,t_0+T} & \vdots & \cdots \\
\vdots & & & \vdots & \vdots & \\
\vdots & & & \vdots & \vdots & \\
q^{(g)}_{x_{\max},t_0} & q^{(g)}_{x_{\max},t_0+1} & \cdots & q^{(g)}_{x_{\max},t_0+T} & \vdots & \cdots
\end{pmatrix}
\begin{array}{l}
\rightarrow \textit{Predictions} \\
\rightarrow \textit{Predictions} \\
\rightarrow \textit{Predictions} \\
\\
\\
\rightarrow \textit{Predictions}
\end{array}
$$

$\leftarrow$ *Observed* $\rightarrow$

$$\hat{q}^{(g)}_{xt} = 1 - \exp\left(-m^{(g)}_{x,t}\right)$$

Model