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Exercises to the lecture

"Advanced Financial Economics I"

Problem Set 1

Due: 16 November 2020, 12:15 pm
Please upload your solution to the OLAT folder or submit it via email to christoph.hambel@hof.uni-frankfurt.de

Problem 1 (Partial Equilibrium)

1. Consider the sequence $Y = (Y_t)_{t \in \mathbb{N}}$ of i.i.d. random variables with

$$p = \mathbb{P}(Y_t = 1) = 1 - \mathbb{P}(Y_t = -1).$$

Denote $X_t = \sum_{j=1}^{t} Y_j$. Show that $M = (M_t)_{t \in \mathbb{N}}$ with $M_t = \left(\frac{1-p}{p}\right)^{X_t}$ is a martingale w.r.t. the natural filtration of Y.

2. Prove the *easy* direction of the Second Fundamental Theorem of Asset Pricing, i.e., show the following: *Suppose the market is free of arbitrage. If the market is complete, there exists a unique equivalent martingale measure* $\mathbb{Q} \sim \mathbb{P}$.

3. Consider an arbitrage-free discrete-time model. One can show that there exists an adapted trading strategy φ (the so-called numeraire portfolio) such that $X_0^\varphi = 1$ and for any asset S^i, the relative price process $\frac{S^i}{X^\varphi}$ is a $\mathbb{P}$-martingale. (i) Explain how the wealth process X^φ relates to the pricing kernel M, the equivalent martingale measure $\mathbb{Q} \sim \mathbb{P}$, and the Radon-Nikodym derivative $Z_T = \frac{d\mathbb{Q}}{d\mathbb{P}}$. (ii) State the numeraire portfolio explicitly in the one-period Binomial model (slide 12).

Problem 2 (CCAPM with quadratic utility)

Consider an economy with infinite time-horizon and N risky assets. There are N trees in the economy and each tree is represented by a single share that entitles the holder to all the output produced. Denote by D_i the dividend (output) produced by asset i. The representative agent in this economy maximizes expected utility of consumption

$$\max_{(C_t)_{t\in\mathbb{N}}} \mathbb{E} \sum_{t=0}^{\infty} \mathrm{e}^{-\delta t} u(C_t)$$

where $u(C_t) = aC_t - \frac{b}{2}C_t^2$, $a, b \in \mathbb{R}$.

1. Derive the first order condition for the optimal investment in each asset and show that the optimal solution is such that

$$1 = \mathbb{E}_t\left[\mathrm{e}^{-\delta} \frac{a - bC_{t+1}}{a - bC_t}(1 + r_{t+1}^i)\right]$$

   where $1+r_{t+1}^i = \frac{D_{t+1}^i + P_{t+1}^i}{P_t^i}$ is the return on asset i over $(t, t+1]$ and $C_t = \sum_{i=1}^N D_t^i$ (market clearing condition with N assets).

2. Determine the pricing kernel and the risk-free rate in this economy.

3. Show that the risk premium of asset i satisfies

$$\mathbb{E}_t[r_{t+1}^i] - r_t^f = \mathrm{e}^{-\delta} \frac{b(1 + r_t^f)}{a - bC_t} \mathrm{cov}_t(C_{t+1}, r_{t+1}^i)$$

and interpret this result.

4. Show that the risk premium of asset i can be rewritten as

$$\mathbb{E}_t[r^i_{t+1}] - r^f_t = \beta^i_t\big(\mathbb{E}_t[r^c_{t+1}] - r^f_t\big)$$

for some β^i_t to be determined and where r^c_{t+1} is the return on the consumption claim.

Problem 3 (Habit Formation) Consider an agent facing the following optimization problem

$$\max_{(C_t)_{t\in\mathbb{N}}} \mathbb{E} \sum_{t=0}^{\infty} \mathrm{e}^{-\delta t} u(C_t, C_{t-1})$$

where $u(C_t, C_{t-1}) = \frac{1}{1-\gamma}\big(\alpha C_t + (1-\alpha)C_{t-1}\big)^{1-\gamma}$ for some constant $\alpha \in (0,1]$. The agent's budget constraint is given by

$$X_{t+1} = (1 + r_t)X_t + Y_t - C_t$$

where $Y_t > 0$ is a stochastic income process.

1. Show that an increase in current consumption decreases marginal utility today.

2. State a reasonable condition such that an increase in current consumption increases marginal utility tomorrow.

3. Derive the Euler equation for the optimal investment and derive the stochastic discount factor.

4. Explain the role of α, i.e., the effect of α on the agent's utility and the stochastic discount factor.

Problem 4 (Campbell and Cochrane 1999) Campbell and Cochrane (1999) impose three conditions (see slide 95): (i) risk-free rate is constant; (ii) habit is predetermined at the steady state $\overline{s}$; (iii) habit is predetermined around the steady state $\overline{s}$.

1. Show that these conditions characterize the sensitivity function $\lambda(s)$ as stated on slide 96.

2. Derive the risk-free rate and the equity premium in this model.

3. Explain the main drivers for the results. Why can this model resolve the equity-premium puzzle?