Part VI

Fixed Income Modeling

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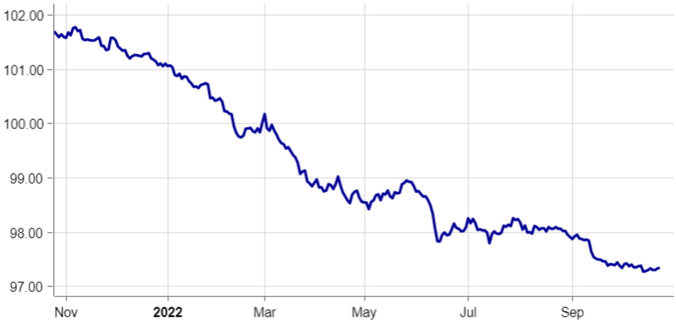
Coupon Bonds

- Now, we are turning to interest rate products beyond a simple money market account.
- Bond = tradeable debt issued by borrower represented by a contract to repay the notional plus interest over the lifetime of the bond.
- Modeling bonds is more involved than modeling stocks because
  1. they pay regular coupons C_i at predefined payment dates $T_i \Rightarrow$ clean vs. dirty prices
  2. they have a finite time horizon T with a known redemption value N
  3. their volatility dies out as $t \to T$
  4. they are exposed to default risk (see Chapter 7) and liquidity risk
- Structure of a coupon bond:

Example: United States of America 1.375% 16/2



- The graph depicts the evolution of the *clean price*.
- The true market price is the *dirty price* = *clean price* + *accrued interest*.
- Accrued interests are paid to compensate the seller for the period during which the bond has been held but for which she will receive no coupon payment.

Example: United States of America 1.375% 16/2

- Bond Volatility is dying out as $t \to T$.
- Clean Price $\to N$ as $t \to T$.
- Dirty Price $\to N + C$ as $t \to T$.

Bond Yields

- The *yield-to-maturity* $y_t^c(T)$ of a coupon bond paying coupons at a rate c ($C = c\Delta_{T_i} N$) and maturing at $T = T_n$ is implicitly defined by

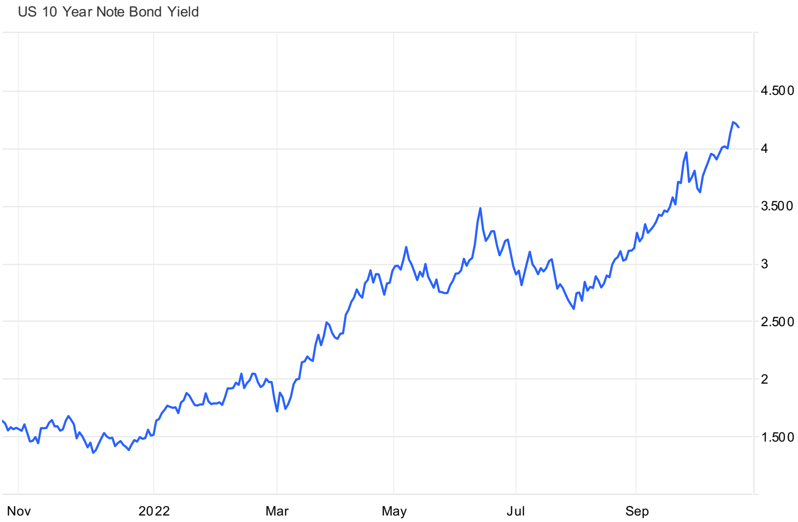
$$P_t^c = \sum_{i=1}^{n} C\mathrm{e}^{-y_t^c(T)(T_i - t)} + N\,\mathrm{e}^{-y_t^c(T)(T-t)}$$

- In practice, bonds are often quoted in terms of yields instead of prices.
- The concept makes the implicit assumption that one can reinvest the coupon payments at the same rate of return.
- Yields of zero-coupon bonds are also called spot rates, i.e., $R_t(T) = y_t^0(T)$.
- Solving for the yield-to-maturity typically requires a computer since closed-form solutions are only available in rare special cases.
- There is an approximation for the *discretely-compounded* yield-to-maturity which admits a nice interpretation:

$$y_{simple} \approx \frac{C}{P_0} + \frac{1}{T-t}\frac{N - P_t}{P_t}$$

First-order Approximation

Evolution of Bond Yields 1y



source: tradingeconomics.com

Evolution of Bond Yields 10y

source: tradingeconomics.com

Evolution of Bond Yields 50y

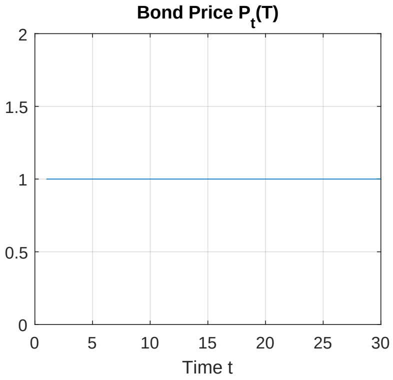
source: tradingeconomics.com

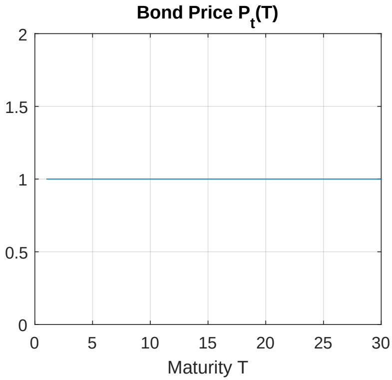
Zero-Coupon Bonds

- A *zero-coupon bond* is a bond that does not pay any coupons.
- A coupon bond is just a portfolio of zero-coupon bonds.
- For our modeling purposes, we consider zero-coupon bonds with notional $N = 1$ only, and assume that these bonds can be traded for every time horizon T. These bonds will be called T*-bonds*.
- The time-t price of a T-bond is denoted by $P_t(T)$. Convention: $P(T) = P_0(T)$.
- This is the discount factor at time t for safe payments made at time T. It represents the "time value of money".
- Arbitrage-free (dirty) price of a coupon bond that pays coupons C at predefined payment dates T_i, $i = 1, \ldots, n$, has a notional N, and matures at time $T = T_n$:

$$P_t^c = \sum_{i=1}^{n} C\, P_t(T_i) + N\, P_t(T)$$

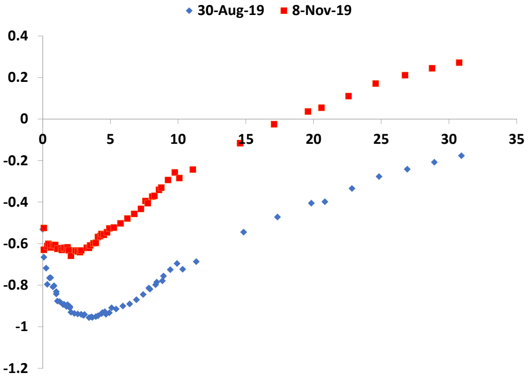
Bond Price versus Discount Curve



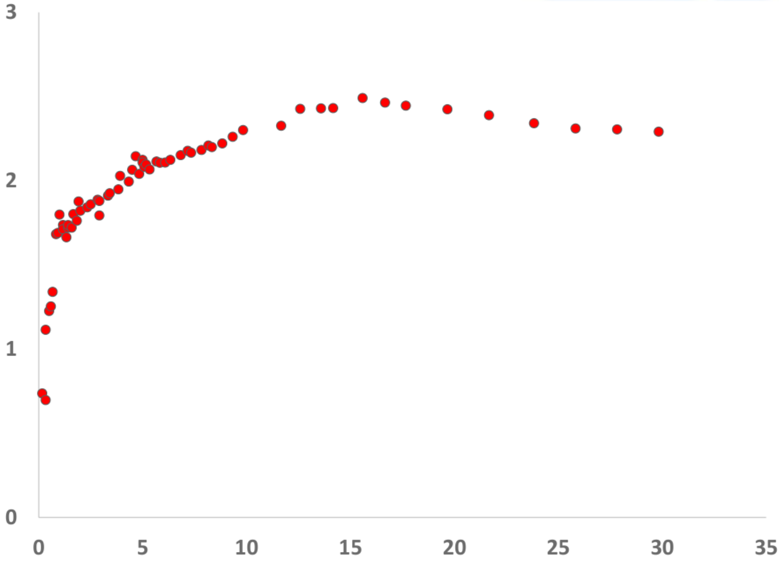


Problem: Term Structure of Interest Rates

1. The graph below depicts the yield curve $T \mapsto y_t(T)$ of German Bundesanleihen in 2019. Plot the yield curve of German Bundesanleihen as of 11th of October (data on Canvas).
2. Explain how and why the term structure has been evolving over the last couple of years and why this might be a problem when we model the term structure of interest rates.



Problem: Solution (1)



Problem: Solution (2)

Outline for Bond Modeling

We have to deal with five problems:

1. **Term Structure of Interest Rates**
   $\rightarrow$ Model how interest rates vary over time.
2. **Coupon Payments**
   $\rightarrow$ Model the prices of zero-coupon bonds. A coupon bond is just a portfolio of zero-bonds.
3. **Finite Time Horizon**
   $\rightarrow$ We already know how to price derivatives with a finite time horizon.
4. **Vanishing Volatility**
   $\rightarrow$ This problem will be solved automatically.
5. **Credit Risk**
   $\rightarrow$ Add a jump process to the dynamics that models credit default (see Chapter 7).

In order to understand how these steps can be carried out we need to establish the relations between interest rates and bond prices.

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Spot Rates vs. Forward Rates

To make discount factors for different maturities more easily accessible, usually a translation is made to interest rates (or *yields to maturity*). There are two fundamental types of interest rates for each bond issuer.

1. **Spot rate** $R_t(T)$ holds at time t for an investment over $[t, T]$. Convention: $R(T) = R_0(T)$.

2. **Forward rate** $F_t(T_1, T_2)$ holds at time t for an investment over $[T_1, T_2]$. Convention: $F(T_1, T_2) = F_0(T_1, T_2)$.

Spot Rates

- The spot rate can be backed out from zero bonds from the equation

$$P_t(T) = \mathrm{e}^{-R_t(T)(T-t)} \quad \Longleftrightarrow \quad R_t(T) = -\frac{1}{T-t} \ln \left(P_t(T)\right)$$

- Price of a coupon bond that pays coupons C at predefined payment dates T_i, $i = 1, \ldots, n$, has a notional N, and matures at time $T = T_n$:

$$P_t^c = \sum_{i=1}^{n} C\mathrm{e}^{-R_t(T_i)(T_i-t)} + N\,\mathrm{e}^{-R_t(T)(T-t)}$$

- The curve that is obtained by plotting $P_t(T)$ against T is called the *discount curve*, i.e., $T \mapsto P_t(T)$
- The curve that is obtained by plotting $R_t(T)$ against T is called the *spot curve*, i.e., $T \mapsto R_t(T)$

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- *A forward agreement* is a contract that allows an investor to log in today an interest rate for an investment over a future time interval.
- Forward rate $F_t(T_1, T_2)$ holds at time t for an investment over $[T_1, T_2]$. Convention: $F(T_1, T_2) = F_0(T_1, T_2)$.

- No arbitrage implies

$$\underbrace{e^{R_t(T_1)(T_1-t)}}_{=1/P_t(T_1)} e^{F_t(T_1,T_2)(T_2-T_1)} = \underbrace{e^{R_t(T_2)(T_2-t)}}_{=1/P_t(T_2)}$$

- Consequently,

$$\begin{aligned} F_t(T_1, T_2) &= \frac{1}{T_2 - T_1} \ln\left(\frac{P_t(T_1)}{P_t(T_2)}\right) \\ &= \frac{1}{T_2 - T_1}\left[R_t(T_2)(T_2 - t) - R_t(T_1)(T_1 - t)\right] \end{aligned}$$

Instantaneous Forward Rate

- We define the *instantaneous forward rate* as

$$F_t(T) = \lim_{\Delta t \to 0} F_t(T, T + \Delta t)$$

- An application of L'Hospitals rule yields

$$F_t(T) = -\frac{\partial}{\partial T} \ln P_t(T) = -\frac{P_t'(T)}{P_t(T)}$$

- Since $\ln P_t(T) = -R_t(T)(T - t)$, we obtain

$$F_t(T) = R_t(T) + (T - t)\,\frac{\partial}{\partial T} R_t(T)$$

- The curve that is obtained by plotting $F_t(T)$ against T is called the *forward curve*, i.e., $T \mapsto F_t(T)$

- The discount factors can be expressed in terms of the forward rates

$$P_t(T) = \mathrm{e}^{-\int_t^T F_t(s)ds}$$

- In particular, to ensure that discount factors are monotonically decreasing it is necessary and sufficient that the forward rates are positive.
- We can express the spot rate in terms of the forward rate by

$$R_t(T) = \frac{1}{T-t}\int_t^T F_t(s)\mathrm{d}s$$

- This shows that the spot rates can be viewed as a cumulative average of the forward rates.

- By definition

$$r_t = \lim_{\Delta t \to 0} R_t(t + \Delta t) = - \lim_{\Delta t \to 0} \frac{\partial}{\partial T} \ln P_t(t + \Delta t) = F_t(t)$$

- A zero-bond with maturity at T can be considered as a "derivative" with constant payoff 1 at T, i.e.,

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{M_t}{M_T} \cdot 1\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]$$

- We thus need appropriate models for the short rate. From these, we can derive
  - (Zero)-coupon bond prices
  - Term structure of interest rates, i.e., the mapping $T \to R_t(T)$
  - Prices of interest rate derivatives

LIBOR Rates

- London Interbank Offered Rate (LIBOR) is an interest-rate average calculated from estimates submitted by the leading banks in London.
- The real-world LIBOR rates are simple interest rates without compounding during their lifetime with maturity in 1 day, 1 month, 3 months, 6 months, and 12 months.
- In this lecture, we refer to LIBOR as a set of discretely compounded risk-free rates.

- Tenor: $\Delta_{T_i} = T_{i+1} - T_i$
- current LIBOR-spot rate for $[t, T_i]$: $L_t(t, T_i)$
- current LIBOR-forward rate for $[T_i, T_j]$: $L_t(T_i, T_j)$
- future LIBOR-spot rate for $[T_i, T_j]$: $L_{T_i}(T_i, T_j)$, $T_i > t$

LIBOR Rates

- Under no arbitrage, the LIBOR-forward rates satisfy

$$1 + L_t(T_i, T_j)(T_j - T_i) = e^{F_t(T_i,T_j)(T_j - T_i)} = \frac{P_t(T_i)}{P_t(T_j)}$$

$$\implies \qquad L_t(T_i, T_j) = \frac{1}{T_j - T_i}\left[\frac{P_t(T_i)}{P_t(T_j)} - 1\right].$$

- Using $\Delta_{T_i} = T_{i+1} - T_i$, the one-period LIBOR-forward rates satisfy

$$L_t(T_i) = L_t(T_i, T_{i+1}) = \frac{1}{\Delta_{T_i}}\left[\frac{P_t(T_i)}{P_t(T_{i+1})} - 1\right]$$

- LIBOR-spot rates:

$$L_{T_i}(T_i, T_j) = \frac{1}{T_j - T_i}\left[\frac{1}{P_{T_i}(T_j)} - 1\right]$$

and the corresponding one-period rate

$$L_{T_i} = L_{T_i}(T_i, T_{i+1}) = \frac{1}{\Delta_{T_i}}\left[\frac{1}{P_{T_i}(T_{i+1})} - 1\right]$$

.

Floating Rate Notes

- A *Floating Rate Note* is a bond with variable coupon payments that are typically linked to a reference rate.
- It is very common in quantitative finance to use LIBOR rates as reference interest rates.
- Variable coupon payments made at times T_i, $i = 1, \ldots, n$ with $\Delta_{T_i} = T_{i+1} - T_i$, are spot LIBOR payments $L_{T_{i-1}} = L_{T_{i-1}}(T_{i-1}, T_i)$ fixed at the *previous* payment date T_{i-1}.
- Payment structure of a FRN:

t	T_1	T_2	$\ldots$	T_{n-1}	$T = T_n$
C_t	$L_{T_0}\Delta_{T_0}N$	$L_{T_1}\Delta_{T_1}N$	$\ldots$	$L_{T_{n-2}}\Delta_{T_{n-2}}N$	$(1 + L_{T_{n-1}}\Delta_{T_{n-1}})N$

Price of a Floating Rate Note

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- Determine the price of the FRN at time T_{n-1}:

$$P^{float}_{T_{n-1}} = P_{T_{n-1}}(T_n)N(1 + L_{T_{n-1}}\Delta_{T_{n-1}}) = \frac{N(1 + L_{T_{n-1}}\Delta_{T_{n-1}})}{1 + L_{T_{n-1}}\Delta_{T_{n-1}}}$$

$$\Longrightarrow \quad P^{float}_{T_{n-1}} = N.$$

- Determine $P^{float}_{T_{n-2}}$ by discounting value components at T_{n-1}
  - value of remaining cash flows: N
  - coupon: $L_{T_{n-2}}N$

discounting yields

$$P^{float}_{T_{n-2}} = \frac{N(1 + L_{T_{n-2}}\Delta_{T_{n-2}})}{1 + L_{T_{n-2}}\Delta_{T_{n-2}}}$$

$$\Longrightarrow \quad P^{float}_{T_{n-2}} = N.$$

Therefore (mathematical induction): $P^{float}_{T_j} = N$. One can also show $P^{float}_{t} = N$ for all $t \leq T$.

Interest Rate Swap

- An interest rate swap is a derivative contract which exchanges one stream of future interest payments for another stream based on a specified principal amount. Interest rate swaps usually involve the exchange of a fixed interest rate $s(T)$ for a floating rate L_t.

- How should the *par swap rate* $s(T)$ be chosen such that the price of the contract is zero at initiation?
- An interest rate swap is equivalent to the exchange of the coupon payments (but not the notionals) of a coupon bond against those of a floating rate note.

Par Swap Rate

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- The swap rate must be chosen such that both products have the same price

$$\underbrace{P_0^s(T)}_{\text{Price of a Coupon bond}} \overset{!}{=} \underbrace{N}_{\text{Price of a FRN}}$$

- Choose $s(T)$ such that the market is free of arbitrage, i.e.,

$$N = \sum_{i=1}^{n} s_0(T)\Delta_{T_{i-1}} N P_0(T_i) + N P_0(T)$$

$$\implies \quad 1 = \sum_{i=1}^{n} s_0(T)\Delta_{T_{i-1}} P_0(T_i) + P_0(T)$$

$$\implies \quad s_0(T) = \frac{1 - P_0(T)}{\sum_{i=1}^{n} \Delta_{T_{i-1}} P_0(T_i)}$$

- The mapping $T \mapsto s_t(T)$ is the *swap curve* at time t.

Value of a Swap

- While the par swap rate $s_0(T)$ is chosen such that the value of the swap at initiation is zero, the swap value will be changing over time.
- We denote the time-t value of a payer swap (i.e., holder is the counterparty that pays the fixed interest) by V_t^{payer}. By construction $V_0^{payer} = 0$.
- If $t > 0$, the value of this swap equals the difference between the floating leg and the fixed leg, i.e.,

$$
\begin{aligned}
V_t^{payer} &= V_t^{float} - V_t^{fixed} \\
&= N[1 - P_t(T)] - s_0(T) \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i) \\
&= s_t(T) \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i) - s_0(T) \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i)
\end{aligned}
$$

- Consequently, the value of a payer swap is

$$V_t^{payer} = [s_t(T) - s_0(T)] \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i)$$

- The value of a receiver swap (holder pays variable interest) at time t is just $V_t^{receiver} = -V_t^{payer}$.
- **Moral**: Swaps can be priced without an interest rate model. All we need is the empirically observable discount curve, i.e., prices of zero-coupon bonds.
- A payer *swaption* is a contract that entitles the holder to enter, at a given time in the future, a payer swap with a specified duration and a swap rate that is determined in advance (the strike).
- To price swaptions, we need a model that describes the evolution of the swap curve over time. $\longrightarrow$ Swap Market Model.

Bond Options

- A European bond option is a contract between two counterparties, whereby the buyer (holder) has the right to buy (Call option) or to sell (Put option) the underlying bond from/to the seller (stillholder) at a predetermined strike price K at its maturity T_1.
- Option with maturity in T_1 on a zero bond with maturity in $T_2 > T_1$:

$$\begin{aligned} Call_{T_1}(P_{T_1}(T_2)) &= (P_{T_1}(T_2) - K)^+ \\ Put_{T_1}(P_{T_1}(T_2)) &= (K - P_{T_1}(T_2))^+ \end{aligned}$$

- Put-call-parity for European bond options

$$Put_t = Call_t - P_t(T_2) + K \cdot P_t(T_1).$$

- To price bond options, we need a model that describes the evolution of the bond prices over time. $\longrightarrow$ Short Rate models, HJM framework.

Interest Rate Options

- Interest rate options are options where the underlying is an interest rate.
- If the underlying interest rate exceeds (caplet) or falls below (floorlet) a certain boundary at maturity, the holder of the option can claim an interest payment.
- Caplet with maturity T_i and strike rate L_C on a notional N has payoff at time T_i:

$$(L_{T_{i-1}} - \underbrace{L_C}_{\text{strike}})^+ \Delta_{T_{i-1}} N$$

- Cap: Portfolio of caplets
  $\Rightarrow$ hedge against increasing interest rates
- Floor: Portfolio of floorlets with payoffs $(L_F - L_{T_{i-1}})^+ \Delta_{T_{i-1}} N$
  $\Rightarrow$ hedge against decreasing interest rates.
- To price swaptions, we need a model that describes the evolution of the LIBOR rates over time. $\longrightarrow$ LIBOR Market Model.

Relation between Interest Rate Options and Swaps

- While an interest rate swap provides a perfect hedge against fluctuating interest rates, a caplet only insures against rising interest rates and a floorlet against shrinking interest rates.
- Consider a long-short portfolio of caplets and floorlets with identical strike rates $\overline{L} = L_C = L_F$:

$$
\begin{aligned}
&\left[(L_{T_{i-1}} - \overline{L})^{+} - (\overline{L} - L_{T_{i-1}})^{+}\right]\Delta_{T_{i-1}} N \\
=&\left[\max(L_{T_{i-1}}, \overline{L}) - \overline{L} - \max(L_{T_{i-1}}, \overline{L}) + L_{T_{i-1}}\right]\Delta_{T_{i-1}} N \\
=&\left[L_{T_{i-1}} - \overline{L}\right]\Delta_{T_{i-1}} N \\
=&L_{T_{i-1}}\Delta_{T_{i-1}} N - \overline{L}\Delta_{T_{i-1}} N
\end{aligned}
$$

- This is identical to an exchange of a variable interest rate and a fixed interest rate, i.e., a one-period interest rate swap.
- Interest rate swaps can thus be decomposed into a long-short portfolio of caps and floors. "Cap – Floor = Payer Swap"

- Caplet with maturity T_i and strike rate L_C on a notional N has payoff at time T_i:

$$
\begin{aligned}
&(L_{T_{i-1}} - L_C)^+ \Delta_{T_{i-1}} N \\
=&\Big(\frac{1}{\Delta_{T_{i-1}}}\big[\frac{1}{P_{T_{i-1}}(T_i)} - 1\big] - L_C\Big)^+ \Delta_{T_{i-1}} N \\
=&\Big(\frac{1}{P_{T_{i-1}}(T_i)} - 1 - \Delta_{T_{i-1}} L_C\Big)^+ N
\end{aligned}
$$

- The caplet value at the fixing date T_{i-1} is

$$
\Big(1 - P_{T_{i-1}} - P_{T_{i-1}} \Delta_{T_{i-1}} L_C\Big)^+ N = \Big(N - P_{T_{i-1}}(1 + \Delta_{T_{i-1}} L_C) N\Big)^+
$$

- A caplet can be viewed as a put option on a zero-coupon bond that matures at time T_i with face value $(1 + \Delta_{T_{i-1}} L_C) N$. The expiry date of the option is T_{i-1}, and the strike is N.

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Modeling the Term Structure of Interest Rates

- We first consider default-free (and perfectly liquid) bonds corresponding to the discount factors and interest rates.
- We start with the benchmark no arbitrage Vasicek model.
- We then generalize this benchmark model, focusing on so-called affine term structure models.
- We will also study the Heath-Jarrow-Morton framework and the LIBOR market model.
- The pricing of bonds can be influenced significantly by credit risk (and liquidity risk) $\longrightarrow$ Chapter 7.

Wishlist for Term Structure Models

A good term-structure model should be able to

- reproduce the currently observed term structure (i.e., bond prices).
- reproduce currently observed prices of other term structure products.
- generate (under $\mathbb{P}$) reasonable future term structures (for instance does not generate (very) negative interest rates).
- capture volatilities of rates for different maturities and correlations between them.
- be tractable; allows quick pricing of popular term structure derivatives such as swaptions and interest rate caps.

Short Rate Models

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- A generic short-rate model for the evolution of the term structure can be written as follows:

$$dX_t = \mu_X(t, X_t)dt + \sigma_X(t, X_t)dW, \qquad r_t = h(t, X_t)$$

- Money Market Account: $dM_t = M_t r_t dt$
- A T-bond is just a derivative with constant payoff $P_T(T) = 1$ at maturity T. Pricing under $\mathbb{Q}$:

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{M_t}{M_T} \cdot 1\right] = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]$$

- The TSIR is thus given by

$$R_t(T) = -\frac{1}{T-t} \log \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]$$

Vasicek model under $\mathbb{P}$ and under $\mathbb{Q}$

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- Vasicek (1977) originally chosed an Ornstein-Uhlenbeck process for the short rate under $\mathbb{P}$:

$$dr_t = a(b - r_t)\,dt + \sigma\,dW_t, \qquad dM_t = M_t r_t dt$$

- This model ($X = r$, $Y = M$) satisfies the NA criterion and λ can be chosen arbitrarily.
- *Assuming* that the market price of risk associated to W_t is a constant λ yielding $dW_t^{\mathbb{Q}} = \lambda\,dt + dW_t$ (where $W_t^{\mathbb{Q}}$ is a BM under $\mathbb{Q}$), and

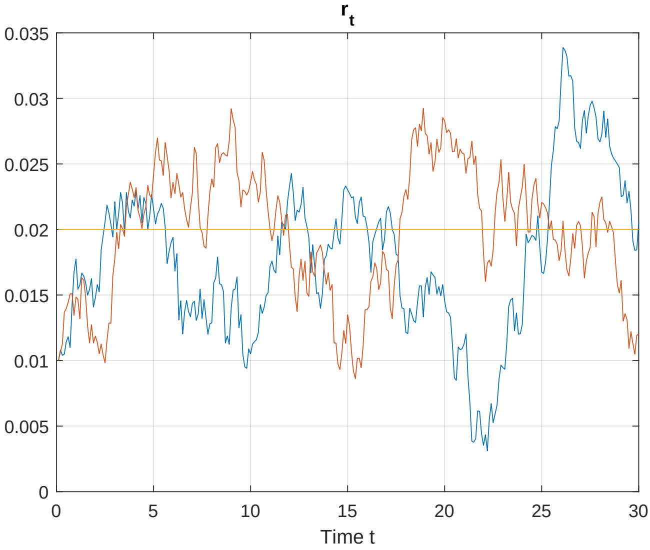
$$dr_t = [a(b - r_t) - \sigma\lambda]\,dt + \sigma\,dW_t^{\mathbb{Q}}$$

which can be written in the form

$$dr_t = a(b^{\mathbb{Q}} - r_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \qquad b^{\mathbb{Q}} = b - \lambda\,\frac{\sigma}{a}\,.$$

- This is the model under $\mathbb{Q}$ as we used it before.

"Typical" Paths of the Vasicek Model



- Show the following properties of the Ornstein-Uhlenbeck process $\mathrm{d}X_t = a(b - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$:

1. $X_t = X_0 \mathrm{e}^{-at} + b(1 - \mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-a(t-s)} \mathrm{d}W_s$

2. $X_t \sim \mathcal{N}\left(\mu(X_t), \sigma(X_t)^2\right)$ with

   $\mu(X_t) = X_0 \mathrm{e}^{-at} + b(1 - \mathrm{e}^{-at})$ and $\sigma^2(X_t) = \frac{1-\mathrm{e}^{-2at}}{2a}\sigma^2$

- **Solution**:

Problem: Solving Ornstein-Uhlenbeck

Problem: Solving Ornstein-Uhlenbeck

- We know that the Vasicek model is free of arbitrage, hence we can formulate it under $\mathbb{Q}$:

$$\mathrm{d}r_t = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}}, \qquad b^{\mathbb{Q}} = b - \lambda\,\frac{\sigma}{a}\,.$$

- We know that the price of a T-bond is just a derivative with constant payoff $P_T(T) = 1$ at maturity T. Pricing under $\mathbb{Q}$:

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{M_t}{M_T}\cdot 1\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]$$

- Question: What would be the pricing relation under $\mathbb{P}$?

- We first calculate $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{M_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right]$.

Bond Price in the Vasicek Model

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- Short rate dynamics: $\mathrm{d}r_t = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}}$
- Dynamics of the log-MMA: $\mathrm{d}\log M_t = r_t \mathrm{d}t$
- Consequently,

$$\mathrm{d}(r_t + a\log M_t) = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}} + a r_t \mathrm{d}t = a b^{\mathbb{Q}}\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}}$$

- Integrating and some algebra yields:

$$\log M_t = \frac{1}{a}\Big[a b^{\mathbb{Q}} t + \sigma W_t^{\mathbb{Q}} - (r_t - r_0)\Big]$$

- We know that $r_t = r_0 \mathrm{e}^{-at} + b^{\mathbb{Q}}(1 - \mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-a(t-s)} \mathrm{d}W_s^{\mathbb{Q}}$. Substituting this solution into $\log M_t$ yields

$$\log M_t = \frac{1}{a}\Big[a b^{\mathbb{Q}} t + \sigma W_t^{\mathbb{Q}} + r_0 - \Big(r_0 \mathrm{e}^{-at} + b^{\mathbb{Q}}(1 - \mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-a(t-s)} \mathrm{d}W_s^{\mathbb{Q}}\Big)\Big]$$

Bond Price in the Vasicek Model

- Therefore, $\log M_t$ follows a normal distribution under $\mathbb{Q}$ with

$$\mathbb{E}^{\mathbb{Q}}[\log M_t] = b^{\mathbb{Q}} t + \frac{1}{a}\left(1 - \mathrm{e}^{-at}\right)\left(r_0 - b^{\mathbb{Q}}\right)$$

$$\begin{aligned} \mathrm{var}^{\mathbb{Q}}[\log M_t] &= \frac{\sigma^2}{a^2}\int_0^t \left[1 - \mathrm{e}^{-a(t-s)}\right]^2 \mathrm{d}s \\ &= \frac{\sigma^2}{a^2}\left[t - \frac{2}{a}\left(1 - \mathrm{e}^{-at}\right) + \frac{1}{2a}\left(1 - \mathrm{e}^{-2at}\right)\right] \end{aligned}$$

- In turn, $-\log M_T$ is normally distributed as well.
- Now, we can calculate $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{M_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right]$, where $\mathrm{e}^{-\log M_T}$ is log-normally distributed, i.e.,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right] = \mathrm{e}^{-\mathbb{E}^{\mathbb{Q}}[\log M_T] + \frac{1}{2} var^{\mathbb{Q}}[\log M_T]}$$

Bond Price in the Vasicek Model

- Substituting everything we know into this expression, we obtain

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-\log M_T}\big] = \exp\Big(- \big[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\big] T - \frac{1 - e^{-aT}}{a}\big[r_0 - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\big]\Big) \cdot \exp\Big(\frac{\sigma^2}{2a^2} \frac{1 - e^{-2aT}}{2a}\Big)$$

- In turn, the current price of a T-bond in the Vasicek model is

$$P_0(T) = \exp\Big(- \big[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\big] T - \frac{1 - e^{-aT}}{a}\big[r_0 - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\big]\Big) \cdot \exp\Big(\frac{\sigma^2}{2a^2} \frac{1 - e^{-2aT}}{2a}\Big)$$

with $b^{\mathbb{Q}} = b - \frac{\sigma\lambda}{a}$.

- The yield curve now follows straightforwardly:

$$R_0(T) = -\frac{1}{T}\log P_0(T)$$
$$= \left[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\right] + \frac{1-\mathrm{e}^{-aT}}{aT}\left[r_0 - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\right] - \frac{\sigma^2}{2a^2}\frac{1-\mathrm{e}^{-2aT}}{2aT}$$

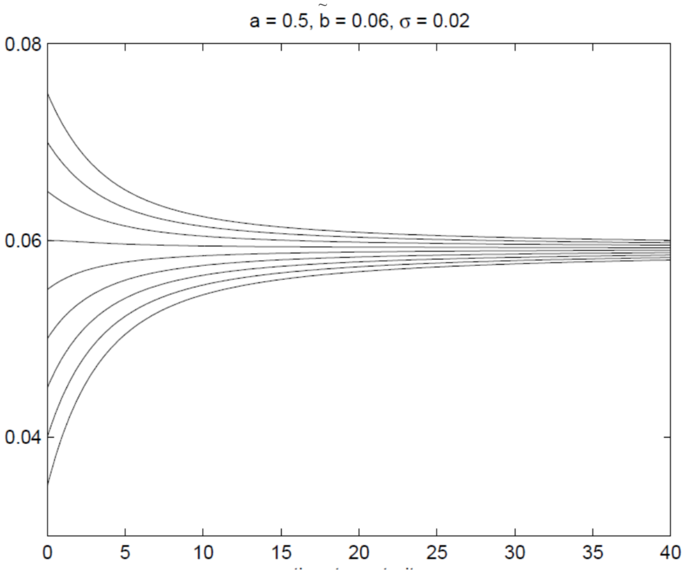
- Taking the limit for super long-term bonds, i.e., $T \to \infty$

$$\overline{R}_0 := \lim_{T\to\infty} R_0(T) = b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}$$

- Therefore,

$$R_0(T) = \overline{R}_0 + \frac{1-\mathrm{e}^{-aT}}{aT}(r_0 - \overline{R}_0) + \frac{\sigma^2}{2a^2}\frac{(1-\mathrm{e}^{-aT})^2}{2aT}$$

The Yield Curve



Empirical Performance

- Due to its normality property the Vasicek model is very tractable both analytically and numerically. In particular, the model can be simulated exactly by the Euler-scheme.
- The empirical performance of the Vasicek model is bad.
  - The current, observed term structure typically is not matched very well, i.e.,

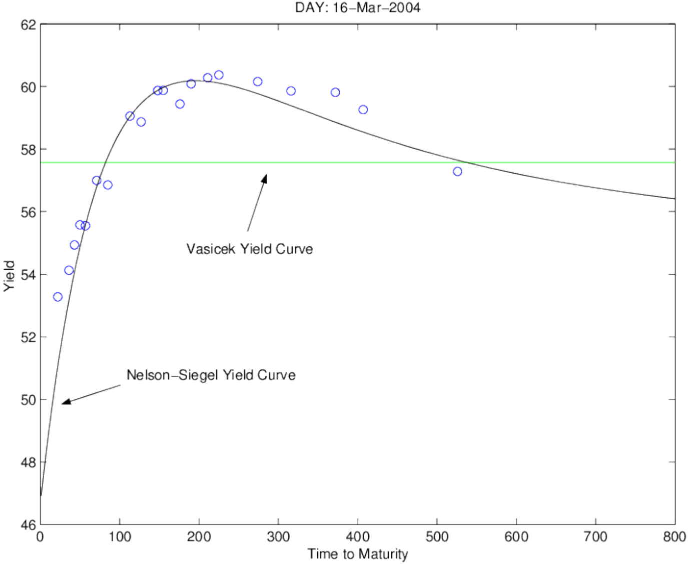
$$R_0(T) = \overline{R}_0 + \frac{1-\mathrm{e}^{-aT}}{aT}(r_0 - \overline{R}_0) + \frac{\sigma^2}{2a^2}\frac{(1-\mathrm{e}^{-aT})^2}{2aT}$$

    is typically not very close to the observed one at time $t = 0$
  - This is particularly pronounced if the term structure has a hump.
- This issue can be addressed by the Hull-White model

$$\mathrm{d}r_t = a(t)\big(b^{\mathbb{Q}}(t) - r_t\big)\,\mathrm{d}t + \sigma(t)\,dW_t^{\mathbb{Q}}$$

Using this approach we can "fit the initial term structure".

Empirical Performance



Remark: (Non)-negativity

- In the Vasicek model, interest rates (yields) can become negative without lower bound.
- This issue can be addressed by the Cox-Ingersol-Ross model

$$\mathrm{d}r_t = a\big(b^{\mathbb{Q}} - r_t\big)\,\mathrm{d}t + \sigma\sqrt{r_t}\,dW_t^{\mathbb{Q}},$$

  which ensures that interest rates stay positive.
- One might want to have negative interest rates, but with a lower bound, e.g.,

$$\mathrm{d}X_t = a\big(b^{\mathbb{Q}} - X_t\big)\,\mathrm{d}t + \sigma\sqrt{X_t}\,dW_t^{\mathbb{Q}}, \qquad r_t = X_t - \ell$$

- The CIR model is much less tractable than the Vasicek model (calculations get much more involved, SDE does not possess an explicit solution, distribution is non-central χ^2, and simulation is challenging).

- We have only studied the case $t = 0$, but this procedure also works for $t > 0$.
- We obtain

$$P_t(T) = \exp\Big(- \big[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\big](T-t) - \frac{1 - \mathrm{e}^{-a(T-t)}}{a}\big[r_t - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\big]\Big) \cdot \exp\Big(\frac{\sigma^2}{2a^2}\frac{1 - \mathrm{e}^{-2a(T-t)}}{2a}\Big)$$

- Consequently, the price can be written as

$$P(t, r; T) = \exp\big(A(t,T) + B(t,T)r_t\big)$$

for functions $A(t,T)$ and $B(t,T) = -\frac{1}{a}\big(1 - \mathrm{e}^{-a(T-t)}\big)$.
- Any short rate model that leads to such a representation of the bond prices will be called an affine short rate model.

- A standard way to estimate the process r_t under $\mathbb{P}$ is to run a regression

$$r_{t+\Delta t} = \alpha + \beta r_t + \varepsilon_{t+\Delta t},$$

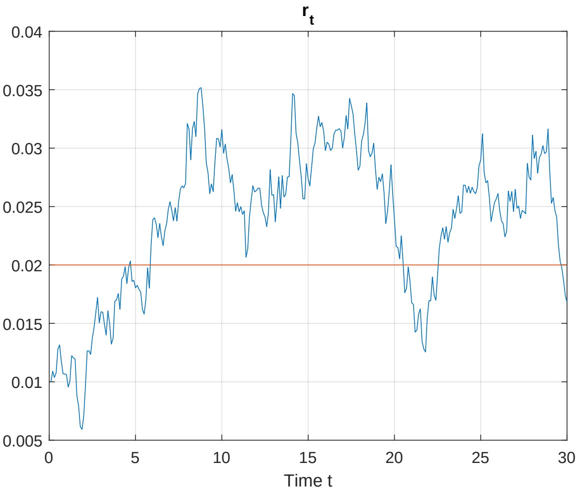
estimated using OLS (under the usual assumptions).

1. What is the link between α, β, and $s^2 = \text{var}(\varepsilon_{t+\Delta t})$ and a, b, and σ?
2. Implement a code that estimates the parameters a, b, and σ for given interest rate data and visualize the regression.
3. Simulate trajectories for the Vasicek model estimated in (2).

- **Solution:** (1)

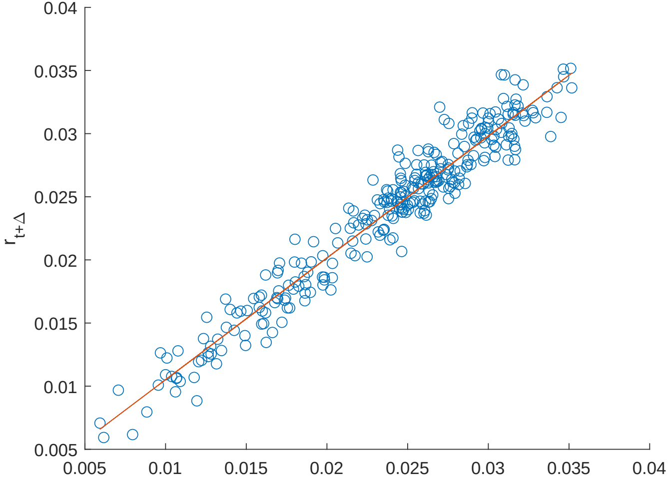
Problem: Estimation of the Vasicek Model

Problem: Data



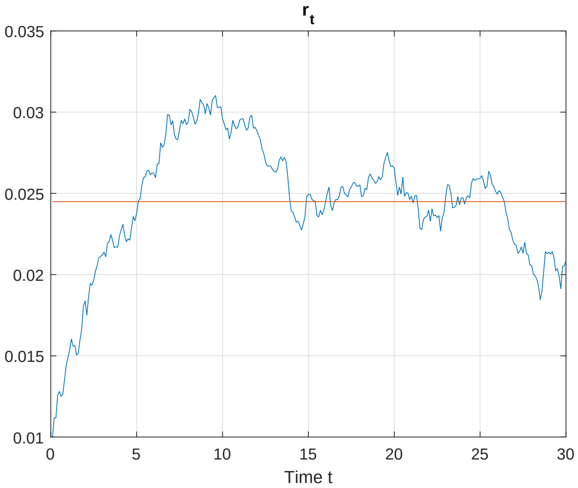
(Simulated) short rate data generated with $r_0 = 0.01$, $a = 0.25$, $b = 0.02$, $\sigma = 0.015$, $\Delta t = 0.1$.

Problem: OLS Regression (2)



Simulated model with $\widehat{\alpha} = 8.6844e - 04$, $\widehat{\beta} = 0.9645$, $\widehat{s} = 0.015$

Problem: Simulation (3)



Regression line with $\widehat{a} = -\frac{\log \widehat{\beta}}{\Delta t} = 0.3610$, $\widehat{b} = \frac{\widehat{\alpha}}{1-\widehat{\beta}} = 0.0245$,
$\sigma = \widehat{s}\sqrt{2\widehat{a}/(1-\mathrm{e}^{-2\widehat{a}\Delta t})} = 0.005$.

- One obvious drawback of the Vasicek model is that it in general does not match observed bond prices. We describe a way to mend this which actually can be applied to *any* term structure model.
- Consider a term structure model of the general form

$$
\begin{aligned}
dX_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\
r_t &= h(t, X_t).
\end{aligned}
$$

- Suppose that the forward curve at current time 0 as produced by the model ($F_0^{un}(T)$; "un" for "unadjusted") does not match the observed forward curve ($F_0^{obs}(T)$). Modify the model as follows:

$$
\begin{aligned}
dX_t &= \mu_X(t, X_t)\,dt + \sigma_X(t, X_t)\,dW_t \\
r_t &= h(t, X_t) + F_0^{obs}(t) - F_0^{un}(t).
\end{aligned}
$$

- Now the model does match the observed forward curve, and hence also the spot yield curve.

Simplest example

- The simplest term structure model is the one in which the short rate is constant: $r_t = r$. The forward curve is given in this case by

$$F_0^{un}(T) = -\frac{\mathrm{d}}{\mathrm{d}T} \log P_0(T) = -\frac{\mathrm{d}}{\mathrm{d}T} \log e^{-rT} = r.$$

- Using the recipe described on the previous slide, we can modify the model so that it matches the current term structure. The modified short rate model is:

$$r_t = F_0^{obs}(t).$$

- This is still a deterministic model. It matches currently observed bond prices. But it will not match the prices of swaptions, for instance.

Adjusting the Vasicek model

- Now take the Vasicek model (under $\mathbb{Q}$)

$$\mathrm{d}r_t = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}.$$

- The corresponding forward curve at time 0 is

$$F_0^{un}(r_0, T) = e^{-aT} r_0 + \left(1 - e^{-aT}\right) b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2.$$

- The modified version that matches the current term structure is (rename the original r_t to X_t)

$$dX_t = a(b^{\mathbb{Q}} - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}$$

$$r_t = X_t + F_0^{obs}(t) - F_0^{un}(X_0, t).$$

Rewrite the model

- The modified Vasicek model can be rewritten by taking the differential of r_t:

$$
\begin{aligned}
\mathrm{d}r_t &= \mathrm{d}X_t + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{obs}(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)\,\mathrm{d}t \\
&= a(b^{\mathbb{Q}} - X_t)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{obs}(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}} \\
&= a\big(b^{\mathbb{Q}} - r_t + F_0^{obs}(t) - F_0^{un}(t)\big)\,\mathrm{d}t \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{obs}(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}.
\end{aligned}
$$

- To compute $aF_0^{un}(t) + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)$, use:

$$
\left(a + \frac{\mathrm{d}}{\mathrm{d}t}\right)(e^{-at}) = 0.
$$

Result: Hull-White model

- From $F_0^{un}(T) = e^{-aT} r_0 + \left(1 - e^{-aT}\right) b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2$ we get

$$aF_0^{un}(t) + \frac{\mathrm{d}}{\mathrm{d}t} F_0^{un}(t) = ab^{\mathbb{Q}} - \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

- The modified Vasicek model becomes

$$\mathrm{d}r_t = \left(\theta(t) - ar_t\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}$$

with

$$\theta(t) = aF_0^{obs}(t) + \frac{\mathrm{d}}{\mathrm{d}t} F_0^{obs}(t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

- This is known as the *one-factor Hull-White model*.

- A term structure model is said to be affine if the yield curves that it produces are of the form

$$R_t(T) = \alpha(t, T) + \beta(t, T)' X_t$$

or equivalently,

$$P_t(T) = \mathrm{e}^{A(t,T)+B(t,T)' X_t}$$

with $\alpha(t, T) = -\frac{A(t,T)}{T-t}$, $\beta(t, T) = -\frac{B(t,T)}{T-t}$

- Notation:

$$\begin{aligned} \alpha(t, T) &: \text{ scalar} \\ \beta(t, T) &: \text{ vector of length } n \\ X_t &: n\text{-dimensional process of state variables at time } t \end{aligned}$$

Examples for Affine Term Structure Models

- A sufficient condition for a model to be affine is

$$\mathrm{d}X_t = \big(\tilde{A}(t)X_t - g(t)\big)\mathrm{d}t + \tilde{B}(X_t)\mathrm{d}W_t^{\mathbb{Q}}, \qquad r_t = h(t)'X_t$$

- Notation:

$$\begin{aligned} X_t &: n\text{-dimensional process of state variables at time } t \\ \tilde{A}(t) &: n \times n\text{-matrix} \\ \tilde{B}(X_t) &: n \times k \text{ matrix such that } \tilde{B}(X_t)\tilde{B}(X_t)' \text{ is affine in } X_t \\ g(t), h(t) &: \text{ vectors of length } n \\ W^{\mathbb{Q}} &: k\text{-dimensional standard Brownian motion under } \mathbb{Q} \end{aligned}$$

- Examples ($r_t = X_t$):
  - Black-Karasinski: $\mathrm{d}(\log X_t) = a(b_t^{\mathbb{Q}} - \log X_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}$
  - CIR: $\mathrm{d}X_t = a(b^{\mathbb{Q}} - X_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t^{\mathbb{Q}}$
  - Dothan: $\mathrm{d}X_t = X_t(a^{\mathbb{Q}}\mathrm{d}t + \sigma\mathrm{d}W_t^{\mathbb{Q}})$
  - Ho-Lee: $\mathrm{d}X_t = \sigma^2 t\mathrm{d}t + \sigma\mathrm{d}W_t^{\mathbb{Q}}$
  - Vasicek / Hull White: $\mathrm{d}X_t = a(b^{\mathbb{Q}} - X_t)\mathrm{d}t + \sigma\mathrm{d}W_t^{\mathbb{Q}}$

Term Structure Equation

- Remember that bond prices are contingent claims on the short rate with terminal value of 1.
- Let $p(t, X; T)$ denote the time-t price of a T-bond. It follows from the Feynman Kac Theorem that bond prices satisfy the following PDE

$$\frac{\partial p}{\partial t} + \nabla p \cdot (\tilde{A}X - g) + \tfrac{1}{2}\mathrm{tr}\Big(H_p\, \tilde{B}(X)\tilde{B}(X)'\Big) = (h'X)p$$

s.t. $p(T, X; T) = 1$

- Since the model is affine, we can rewrite $\tilde{B}(X)\tilde{B}(X)' = \tilde{C} + \tilde{D}\,X$.

$$\frac{\partial p}{\partial t} + \nabla p \cdot (\tilde{A}X - g) + \tfrac{1}{2}\mathrm{tr}\Big(H_p(\tilde{C} + \tilde{D}\,X)\Big) = (h'X)p$$

- In affine models, bond prices are given by

$$p(t, X; T) = \mathrm{e}^{A(t,T) + B(t,T)'X_t}$$

that can be substituted into the TSE yielding ODEs for A and B s.t. $A(T, T) = B(T, T) = 0$.

Example: Vasicek Revisited

- The TSE is given by

$$\frac{\partial p(t,r;T)}{\partial t} + \frac{\partial p(t,r;T)}{\partial r} a(b^{\mathbb{Q}} - r) + \frac{1}{2}\frac{\partial^2 p(t,r;T)}{\partial r^2}\sigma^2 = p(t,r;T)r$$

- Substituting the conjecture into the TSE

$$p[\dot{A}(t,T) + \dot{B}(t,T)r] + pB(t,T)a(b^{\mathbb{Q}} - r) + \tfrac{1}{2}pB(t,T)^2\sigma^2 = pr$$

- Dividing by p and separating yields

$$\dot{A}(t,T) + B(t,T)ab^{\mathbb{Q}} + \tfrac{1}{2}B(t,T)^2\sigma^2 + r[\dot{B}(t,T) - aB(t,T) - 1] = 0$$

- We obtain two ODEs s.t. $A(T,T) = B(T,T) = 0$:

$$\begin{aligned} \dot{A}(t,T) + B(t,T)ab^{\mathbb{Q}} + \tfrac{1}{2}B(t,T)^2\sigma^2 &= 0 \\ \dot{B}(t,T) - aB(t,T) - 1 &= 0 \end{aligned}$$

Example: Vasicek Revisited

- Linear ODE for B: $\dot{B}(t,T) - aB(t,T) - 1 = 0$ (e.g., Feynman-Kac):

$$B(t,T) = \int_t^T \mathrm{e}^{-a(s-t)}(-1)\mathrm{d}s = -\frac{1}{a}\left(1 - \mathrm{e}^{-a(T-t)}\right)$$

- Integrating A:

$$\begin{aligned} A(t,T) &= \int_t^T B(s,T)ab^{\mathbb{Q}} + \tfrac{1}{2}B(s,T)^2\sigma^2\mathrm{d}s \\ &= \dots \end{aligned}$$

- Bond price as it was before

$$P(t,r;T) = \exp\left(A(t,T) + B(t,T)r_t\right).$$

Example: Cox-Ingersol-Ross

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- The TSE is given by

$$\frac{\partial p(t,r;T)}{\partial t} + \frac{\partial p(t,r;T)}{\partial r} a(b^{\mathbb{Q}} - r) + \frac{1}{2}\frac{\partial^2 p(t,r;T)}{\partial r^2}\sigma^2 r = p(t,r;T)r$$

- Substituting the conjecture into the TSE

$$p[\dot{A}(t,T) + \dot{B}(t,T)r] + pB(t,T)a(b^{\mathbb{Q}} - r) + \frac{1}{2}pB(t,T)^2\sigma^2 r = pr$$

- Dividing by p and separating yields

$$\dot{A}(t,T) + B(t,T)ab^{\mathbb{Q}} + r[\dot{B}(t,T) - aB(t,T) + \frac{1}{2}B(t,T)^2\sigma^2 - 1] = 0$$

- We obtain two ODEs s.t. $A(T,T) = B(T,T) = 0$:

$$\dot{A}(t,T) + B(t,T)ab^{\mathbb{Q}} = 0$$
$$\dot{B}(t,T) - aB(t,T) + \frac{1}{2}B(t,T)^2\sigma^2 - 1 = 0$$

Example: Cox-Ingersol-Ross

- Now, the ODE for B is much more involved, a so-called Riccati equation.

$$\dot{B}(t,T) - aB(t,T) + \frac{1}{2}B(t,T)^2\sigma^2 - 1 = 0$$

- For constant coefficients, by guessing $B(t,T) = k\frac{\Psi_t}{\Psi}$ for a constant k, and a function Ψ, it can be transformed into a linear second-order ODE with well-known solution.
- In the end, we obtain:

$$B(t,T) = -\frac{2(\mathrm{e}^{\gamma(T-t)} - 1)}{\mathrm{e}^{\gamma(T-t)}(\gamma + a) + \gamma - a}, \qquad \gamma = \sqrt{a^2 + 2\sigma^2}$$

- Integrating A:

$$A(t,T) = \int_t^T B(s,T)ab^{\mathbb{Q}}\mathrm{d}s = \frac{2ab^{\mathbb{Q}}}{\sigma^2}\log\left(\frac{2\gamma\mathrm{e}^{0.5(a+\gamma)(T-t)}}{(\gamma + a)(\mathrm{e}^{\gamma(T-t)} - 1) + 2\gamma}\right)$$

- A Hull-White-type extension of the CIR model would make the calculations extremely messy.

Remarks on Option Pricing

- It is also possible to derive closed-form solutions for European call and put options on zero bonds in affine term structure models.
- The option pricing formulas are very similar to the Black-Scholes formula, but we need another EMM to derive them.
- As for bond prices, the option pricing formula for the CIR is significantly more involved than for the Gaussian models.
- We will address this issue in Section 16.
- It is also possible to derive closed-form option prices for claims on the short rate, i.e., options of the form

$$C(T, r_T) = \Phi(r_T).$$

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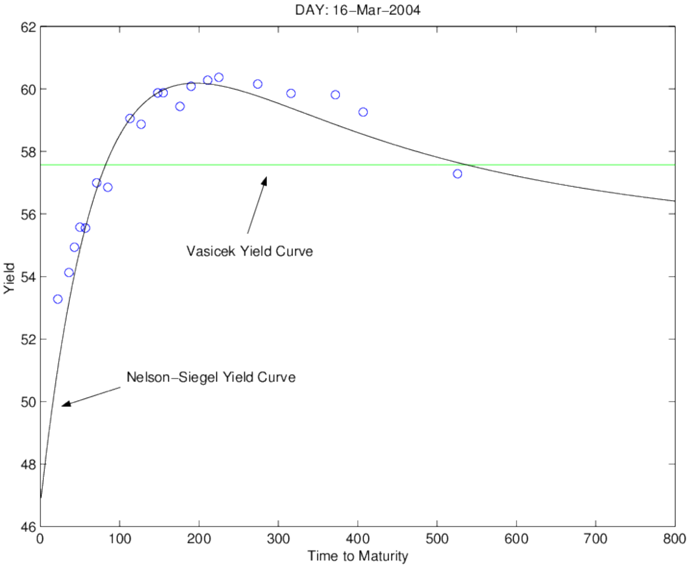
Empirical Performance



Nelson-Siegel Model

- Single-factor short rate models are not sufficient to model the whole TSIR.
- In the Nelson-Siegel model, the term structure is fitted by a deterministic function with four parameters rather than a dynamic short rate.

$$R_t(T) = \beta_{0,t} + \frac{1-\mathrm{e}^{-a_t(T-t)}}{a_t(T-t)}\beta_{1,t} + \left(\frac{1-\mathrm{e}^{-a_t(T-t)}}{a_t(T-t)} - \mathrm{e}^{-a_t(T-t)}\right)\beta_{2,t}$$

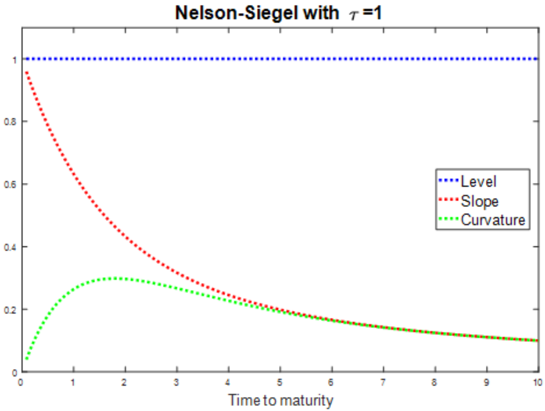
- This implies the forward rate

$$F_t(T) = \beta_{0,t} + \mathrm{e}^{-a_t(T-t)}\beta_{1,t} + a_t(T-t)\mathrm{e}^{-a_t(T-t)}\beta_{2,t}$$

- We use the notation $\tau_t = 1/a_t$.

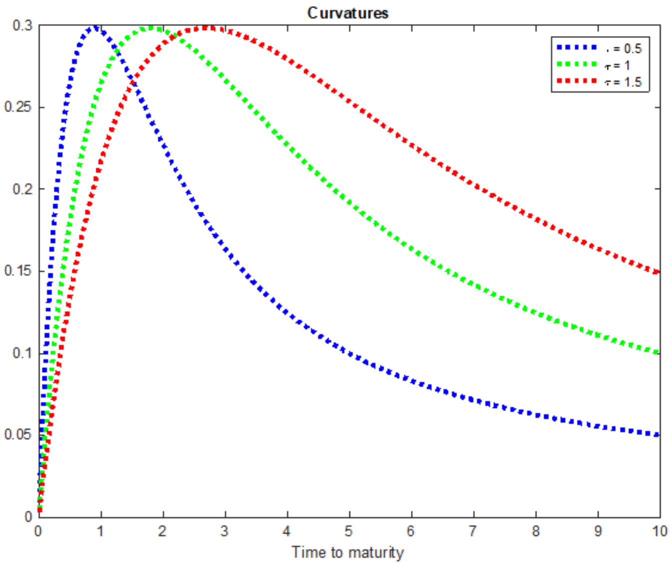
Nelson-Siegel Model

$$R_t(T) = \beta_{0,t}1 + \frac{1 - e^{-a_t(T-t)}}{a_t(T-t)}\beta_{1,t} + \left(\frac{1 - e^{-a_t(T-t)}}{a_t(T-t)} - e^{-a_t(T-t)}\right)\beta_{2,t}$$



$\beta_{0,t}$: long rate, $\beta_{0,t} + \beta_{1,t}$: short rate, $\beta_{2,t}$: size of hump,
$\tau_t = 1/a_t$: determines the time of hump

Nelson-Siegel Model



Dynamic Nelson-Siegel

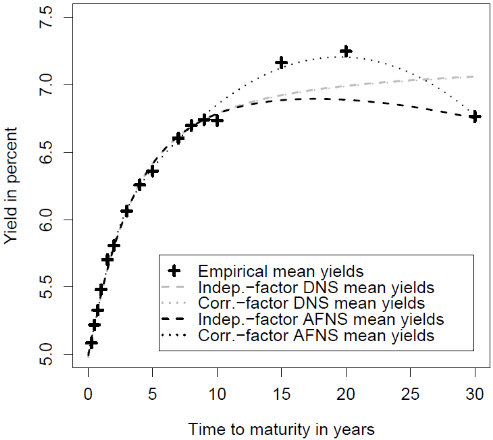
- **Huge drawback**: The Nelson-Siegel term structure cannot be implied by *any* arbitrage-free short-term model.
- **Idea**: Construct a version of the Nelson-Siegel model with factors $\beta_{0,t}, \beta_{1,t}, \beta_{2,t}$ that evolve dynamically over time such that the model reproduces the Nelson-Siegel term structure *as close as possible*.
- Introduce a three-dimensional state process $X_t = (\beta_{0,t}, \beta_{1,t}, \beta_{2,t})'$, and assume

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}}, \qquad r_t = \rho_0(t) + \rho_1(t)'X_t$$

- One can show that for a particular affine parameter choice (see Christensen et al. 2010), the resulting yield curve is

$$R_t(T) = \beta_{0,t} + \frac{1 - \mathrm{e}^{-a(T-t)}}{a(T-t)}\beta_{1,t} + \Big(\frac{1 - \mathrm{e}^{-a(T-t)}}{a(T-t)} - \mathrm{e}^{-a(T-t)}\Big)\beta_{2,t} - \frac{C(t,T)}{T-t}$$

- The resulting model is free of arbitrage, and, due to its affine structure, it has a closed-form solution.
- The empirical performance of this arbitrage-free Nelson-Siegel model (AFNS) is very good.



(Nelson-Siegel-)Svensson Model

- Modification of the Nelson-Siegel Model with six parameters

$$R_t(T) = \beta_{0,t} + \frac{1-e^{-a_t(T-t)}}{a_t(T-t)}\beta_{1,t} + \left(\frac{1-e^{-a_t(T-t)}}{a_t(T-t)} - e^{-a_t(T-t)}\right)\beta_{2,t} + \left(\frac{1-e^{-b_t(T-t)}}{b_t(T-t)} - e^{-b_t(T-t)}\right)\beta_{3,t}$$

- This implies the forward rate

$$F_t(T) = \beta_{0,t} + e^{-a_t(T-t)}\beta_{1,t} + a_t(T-t)e^{-a_t(T-t)}\beta_{2,t} + b_t(T-t)e^{-b_t(T-t)}\beta_{3,t}$$

Dynamic (Nelson-Siegel-)Svensson Model

- Like Nelson-Siegel, also Svensson can be turned into a multi-factor model, with four factors.
- The resulting dynamic Svensson model is also not arbitrage-free (by construction) for *any* short-rate model.
- But, the dynamic *four*-factor Svensson model can also be turned into an arbitrage-free affine *five-factor* term structure model. However, it turns out that this requires the introduction of an extra (slope) factor, together with a non-random correction term.

$$
\begin{aligned}
R_t(T) = \beta_{0,t} &+ \frac{1-e^{-a(T-t)}}{a(T-t)}\beta_{1,t} + \left(\frac{1-e^{-a(T-t)}}{a(T-t)} - e^{-a(T-t)}\right)\beta_{2,t} \\
&+ \frac{1-e^{-b(T-t)}}{b(T-t)}\beta_{4,t} + \left(\frac{1-e^{-b(T-t)}}{b(T-t)} - e^{-b(T-t)}\right)\beta_{3,t} \\
&- \frac{C(t,T)}{T-t}
\end{aligned}
$$

- Given a set of observed bond prices $P_0^{obs}(C, N, T_1, \dots, T_n)$ at time 0.
- Calibrate the six parameters $\pi = \{\beta_0, \beta_1, \beta_2, \beta_3, a, b\}$ such that theoretical prices

$$P_0^{model}(C, N, T_1, \dots, T_n) = \sum_{i=1}^{n} C\mathrm{e}^{-R_0(T_i)T_i} + N\mathrm{e}^{-R_0(T_n)T_n}$$

with

$$R_0(T) = \beta_0 + \frac{1 - \mathrm{e}^{-aT}}{aT}\beta_1 + \left(\frac{1 - \mathrm{e}^{-aT}}{aT} - \mathrm{e}^{-aT}\right)\beta_2 + \left(\frac{1 - \mathrm{e}^{-bT}}{bT} - \mathrm{e}^{-bT}\right)\beta_3$$

closely match the observed prices.

Calibration of the Svensson Model

- This can be achieved by an OLS minimization over the parameter set $\pi = \{\beta_0, \beta_1, \beta_2, \beta_3, a, b\}$:

$$\widehat{\pi} = \arg\min_{\pi} \sum_{j=1}^{J} w_j \big[P_0^{obs,j}(C^j, N^j, T_1^j, \ldots, T_n^j) - P_0^{model,j}(C^j, N^j, T_1^j, \ldots, T_n^j) \big]^2$$

- ECB estimates the six Svensson parameters daily.
- The dynamic versions of those models can be estimated by principal component analysis.

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Motivation

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- So far, we have studied interest rate models where the short rate r is the only explanatory variable.
- Main advantages:
  - Specifying r as the solution of an SDE allows us to use Markov process theory, so we may work within a PDE framework.
  - In particular it is often possible to obtain analytical formulas for bond prices and derivatives.
- Main disadvantages:
  - It is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short rate model.
  - As the short rate model becomes more realistic, the inversion of the yield curve becomes increasingly more difficult.
- Arbitrage-free Nelson-Siegel Models require more state variables. The HJM-framework goes beyond that idea and models the *whole* forward curve.

- The HJM-framework is not a specific model, but a framework for modeling the forward rates.
- We will see that the framework contains the short-rate models as special cases.
- $\mathbb{P}$-dynamics of the forward curve:

$$\mathrm{d}X_t = \mu_X(t, X_t)\mathrm{d}t + \sigma_X(t, X_t)\mathrm{d}W_t, \qquad F_t(T) = h(t, T, X_t)$$
$$r_t = h(t, t, X_t)$$

  where the initial forward curve $F_0(T) = h(0, T, X_0)$ can be observed on the market.
- The HJM framework can, by construction, match the initial term structure.

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- The dynamics of the forward rate follow from Itô's lemma:

$$\mathrm{d}F_t(T) = \mathrm{d}h(t, T, X_t) = \mu_F(t, T, X_t)\mathrm{d}t + \sigma_F(t, T, X_t)\mathrm{d}W_t$$

- Therefore,

$$F_t(T) = F_0(T) + \int_0^t \mu_F(s, T, X_s)\mathrm{d}s + \int_0^t \sigma_F(s, T, X_s)\mathrm{d}W_s$$
$$r_t = F_0(t) + \int_0^t \mu_F(s, t, X_s)\mathrm{d}s + \int_0^t \sigma_F(s, t, X_s)\mathrm{d}W_s$$

- One can show that under $\mathbb{Q}$, the drift terms are fully determined by the specification of the volatility terms $\sigma_F(t, T, X_t)$, and more precisely ...

Heath-Jarrow-Morton Drift Condition

Assume that the induced bond market is arbitrage free. Then there exists a k-dimensional column-vector process $\lambda(t, T, X_t)$ (market price of risk) such that

$$\mu_F(t, T, X_t) = \sigma_F(t, T, X_t) \int_0^t \sigma_F(s, T, X_s)\mathrm{d}s + \sigma_F(t, T, X_t)\lambda(t, T, X_t)$$

- I skip the proof, and focus on the implications:
- $\mathbb{Q}$-dynamics of the forward curve:

$$\begin{aligned} \mathrm{d}F_t(T) &= [\underbrace{\mu_F(t, T, X_t) - \sigma_F(t, T, X_t)\lambda(t, T, X_t)}_{\mu_F^{\mathbb{Q}}(t,T,X_t)}]\mathrm{d}t + \sigma_F(t, T, X_t)\mathrm{d}W_t^{\mathbb{Q}} \\ &= \sigma_F(t, T, X_t)\Big(\int_0^t \sigma_F(s, T, X_s)\mathrm{d}s\Big)\mathrm{d}t + \sigma_F(t, T, X_t)\mathrm{d}W_t^{\mathbb{Q}} \end{aligned}$$

Heath-Jarrow-Morton

- Interest Rates under $\mathbb{Q}$

$$
\begin{aligned}
F_t(T) = F_0(T) &+ \int_0^t \sigma_F(s, T, X_s)\Big(\int_0^s \sigma_F(\tau, T, X_\tau)\mathrm{d}\tau\Big)\mathrm{d}s \\
&+ \int_0^t \sigma_F(s, T, X_s)\mathrm{d}W_s^{\mathbb{Q}} \\
r_t &= F_t(t)
\end{aligned}
$$

- Recipe for the HJM framework:
  1. Specify, by your own choice, the volatilities σ_F.
  2. Determine the drift rate of forward rates under $\mathbb{Q}$: $\mu_F^{\mathbb{Q}}(t, T, X_t) = \sigma_F(t, T, X_t) \int_0^t \sigma_F(s, T, X_s)\mathrm{d}s$.
  3. Go to the market and observe today's forward rate structure $F_0(T)$.
  4. Calculate or simulate the evolution of the term structure $F_t(T)$.
  5. Determine bond prices $P_t(T) = \exp(-\int_t^T F_t(s)\mathrm{d}s)$.
  6. Calculate prices of interest rate derivatives.

Problem: Special Cases

1. Suppose the forward rate volatility is given by $\sigma_F(t, T, X_t) = \sigma$. Show that this specification implies the Ho-Lee model.
2. Suppose the forward rate volatility is given by $\sigma_F(t, T, X_t) = \sigma \mathrm{e}^{-a(T-t)}$. Show that this specification implies the Hull-White model.
3. Show that if $\sigma_F(t, T, X_t)$ is a deterministic function of t and T, all short rates and forward rates are normally distributed. Besides, all bond prices are log-normally distributed.

Solution:

Problem: Special Cases

Problem: Special Cases

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- Since the seminal work of Black (1976) practitioners have been using the Black76-formula for caplets and floorlets.
- A caplet with maturity T_i and strike rate L_C on a notional N has payoff at time T_i:

$$C_{T_i} = (L_{T_{i-1}} - L_C)^+ \Delta_{T_{i-1}} N$$

where $L_{T_{i-1}}$ denotes the spot LIBOR rate for $[T_{i-1}, T_i]$.

- Black (1976) *postulates* the following pricing formula for $t \leq T_{i-1}$:

$$C_t = \Delta_{T_{i-1}} P_t(T_i) L_t(T_{i-1}, T_i) N \cdot \Phi(d_1) - P_t(T_i) \cdot L_C \Delta_{T_{i-1}} N \cdot \Phi(d_2)$$

where d_1 and d_2 are very similar to the terms in the Black-Scholes model.

The T-Forward Measure

- Recall: Numéraire-dependent pricing formula

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right].$$

- We have used
  - $\mathbb{Q}$ associated to the MMA
  - $\mathbb{P}$ associated to the numéraire portfolio
  - $\mathbb{Q}_S$ associated to the stock
- For the pricing of interest rate options, it has proven to be useful to use T-bonds with price $P_t(T)$ as numéraire.
- The corresponding EMM is the so-called T-forward measure $\mathbb{Q}_T$.

$$C_t = P_t(T)E_t^{\mathbb{Q}_T}\left[C_T\right].$$

- This measure disentangles discounting and the calculation of the expectation.

The T-Forward Measure

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$$C_t = P_t(T) E_t^{\mathbb{Q}_T}\left[C_T\right].$$

- This measure disentangles discounting and the calculation of the expectation.

Prove that under $\mathbb{Q}_T$, the instantaneous forward rate $F_0(T)$ is the expected future short rate r_T, i.e.,

$$F_0(T) = \mathbb{E}^{\mathbb{Q}_T}[r_T].$$

Solution:

LIBOR Market Model

- Model the LIBOR forward rates $L_t(T_{i-1}, T_i)$ such that they are log-normally distributed under the T_i-forward measure.
- The LIBOR market model:

$$dL_t(T_{i-1}, T_i) = L_t(T_{i-1}, T_i)\sigma_i(t)' dW_t^{\mathbb{Q}_{T_i}}$$

where $\sigma_i(t) \in \mathbb{R}^k$, $W^{\mathbb{Q}_{T_i}}$ is a k-dimensional Brownian motion.

- Remark: From the definition it is not obvious that, given a specification of $\sigma_i(t)$, there exists a corresponding LIBOR market model. However, it does!
- Idea: Model all LIBOR rates under a common reference measure, the *terminal measure* $\mathbb{Q}_T$ with $T = T_n$

$$dL_t(T_{i-1}, T_i) = \mu_i(t, L_t)dt + L_t(T_{i-1}, T_i)\sigma_i(t) dW_t^{\mathbb{Q}_T}$$

- If one chooses the drift rate appropriately, one obtains the desired LIBOR market specification

$$dL_t(T_{i-1}, T_i) = L_t(T_{i-1}, T_i)\sigma_i(t)' dW_t^{\mathbb{Q}_{T_i}}$$

- One can show that the required drift specification is

$$\mu_i(t, L_t) = -L_t(T_{i-1}, T_i) \sum_{k=i+1}^{n} \frac{\Delta_{T_{k-1}}}{1 + L_t(T_{k-1}, T_k)\Delta_{T_{k-1}}} \sigma_i(t)' \sigma_k(t),$$

$$\mu_n(t, L_t) = 0.$$

- Takeaway: We can model LIBOR rates under the *common* terminal measure $\mathbb{Q}_T$ such that LIBOR forward rates $L_t(T_{i-1}, T_i)$ are log-normally distributed martingales under "*their*" T_i-forward measure $\mathbb{Q}_{T_i}$.

Modeling Choices

- To complete the LIBOR model, one still needs to specify the number k of Brownian motions and the volatilities $\sigma_i(t)$.
- The number k is usually chosen in the range from one to three (correlation does not affect the prices of plain vanilla options, but of more complicated products).
- The volatilities $\sigma_i(t)$ are obtained by calibration to observed price data, i.e., they are *implied* volatilities to match prices of interest rate options. Dependence on time t is often allowed, to ensure sufficient flexibility. $\sigma_i(t)$ is typically a piecewise constant scalar function with jumps at the reset dates.
- Use this calibrated model to determine the prices of more complex products.
- Note: the LIBOR market model does not specify the short rate process and can only price a limited range of term structure products in closed-form.

The Black76 Formula

- Under the T_i-forward measure, the LIBOR forward rate $L_t(T_{i-1}, T_i)$ is a martingale and it is log-normally distributed. Hence, we are in a similar situation as in the Black-Scholes model.
- Straightforward calculations show that the price of a caplet is given by

$$C_t = P_t(T_i)\big[L_t(T_{i-1}, T_i) \cdot \Phi(d_1) - L_C \cdot \Phi(d_2)\big]\Delta_{T_{i-1}} N$$

where

$$d_1 = \frac{\log\left(\frac{L_t(T_{i-1}, T_i)}{L_C}\right) + \frac{1}{2}\Sigma_i(t, T_{i-1})^2}{\Sigma_i(t, T_{i-1})}$$

$$d_2 = d_1 - \Sigma_i(t, T_{i-1})$$

$$\Sigma_i(t, T_{i-1})^2 = \int_t^{T_{i-1}} \|\sigma_i(s)\|^2 \mathrm{d}s$$

Comparison to Black-Scholes

Some Remarks

- There is a one-to-one mapping between the volatility and the caplet price. There is no ambiguity in quoting the price of a caplet simply by quoting its "Black volatility" or implied volatility.
- Caps and floors have the same implied volatility for a given strike.
- As negative interest rates became a possibility, the Black model became increasingly inappropriate. Many variants have been proposed, including shifted log-normal and normal, though a new standard is yet to emerge.
- There is a very general option pricing formula for a European call option with strike K and maturity T on an underlying S. One can show that under mild assumptions the price of a European call option has always the form

$$C_t = S_t \mathbb{Q}^S(S_T > K) - P_t(T) K \mathbb{Q}^T(S_T > K).$$

where $\mathbb{Q}^S$ is an EMM that takes the underlying as numéraire, and $\mathbb{Q}^T$ is the T-forward measure.

General Option Pricing Formula

- This formula holds for *any* arbitrage-free financial market model.
- Suppose the process $\widehat{S}_t = \frac{S_t}{P_t(T)}$ satisfies a stochastic differential equation of the form

$$d\widehat{S}_t = \widehat{S}_t \mu(t, T) dt + \widehat{S}_t \sigma(t, T) dW_t,$$

- Then, the price of the call option is

$$C_t = S_t N(d_1) - P_t(T) K \, N(d_2)$$

with

$$d_1 = \frac{\log\left(\frac{S_t}{K \, P_t(T)}\right) + \frac{1}{2}\Sigma(t, T)^2}{\Sigma(t, T)}$$

$$d_2 = d_1 - \Sigma(t, T)$$

$$\Sigma(t, T)^2 = \int_t^T \|\sigma(s, T)\|^2 ds$$

1. Derive the price of a European call option on a T_2-bond with strike price K and maturity in $T_1 < T_2$ in the Hull-White model,

$$\mathrm{d}r_t = a(b^{\mathbb{Q}}(t) - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}}$$

2. Explain the differences between your result and the option price in the Vasicek model.

Solution:

Problem: Option Pricing in the Hull-White Mode

Swap Market Model

- The *swap market model* is a variant of the LIBOR market model.
- In the swap market model, par swap rates are modeled to be log-normally distributed, rather than LIBOR rates.
- The swap market model is commonly used to price swaptions, i.e., options on swap contracts, for which a variant of the Black76 formula exists.
- It can be shown that LIBOR market models and swap market models are incompatible, i.e., par swap rates are not log-normally distributed in the LIBOR market model, and LIBOR rates are not log-normally distributed in swap market models.