Part II

Generic State Space Model

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Generic State Space Model

- We consider a general framework with n state variables and m assets
- The state variables may include asset prices (in this case $X_i = Y_i$) such as
  - Bonds
  - Commodities
  - Money market account
  - Stocks
  - ...
- But they can also model non-tradable financial or economic factors, such as
  - Interest rates
  - Volatility
  - Expected rate of return
  - Inflation
  - GDP growth
  - ...
- The model is driven by k risk sources (Brownian motions).

Generic State Space Model

- General continuous-time financial market model driven by Brownian motion:

Generic State Space Model

$$
\begin{aligned}
\mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\
Y_t &= \pi_Y(t, X_t).
\end{aligned}
$$

- Notation:

$$
\begin{aligned}
W_t &: k\text{-dimensional standard Brownian motion} \\
X_t &: n\text{-dimensional Markov process of state variables} \\
Y_t &: m\text{-dimensional process of asset prices at time } t \\
\mu_X(t, X_t) &: \text{vector of length } n \\
\sigma_X(t, X_t) &: \text{matrix of size } n \times k \\
\pi_Y(t, X_t) &: \text{vector of length } m \\
t &: \text{time, measured in years}
\end{aligned}
$$

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- Given the functions μ_X, σ_X, and π_Y, we can determine the asset dynamics $\mathrm{d}Y$ on the basis of Itô's lemma.
- Fix a component $C = Y_i$ ("claim") for some $i = 1, \ldots, m$ from the vector of asset prices $Y = (Y_1, \ldots, Y_m)'$.
- Define the real function $\pi_C = \pi_{Y,i}$. Itô's lemma yields (see slide 31).

$$\mathrm{d}C_t = \mu_C(t, X_t)\,\mathrm{d}t + \sigma_C(t, X_t)\,\mathrm{d}W_t$$

with

$$\begin{aligned}
\mu_C &= \frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X + \frac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X \sigma_X'\Big) \\
&= \frac{\partial \pi_C}{\partial t} + \sum_{i=1}^{n} \frac{\partial \pi_C}{\partial x_i}\mu_{X,i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\ell=1}^{k} \frac{\partial^2 \pi_C}{\partial x_i \partial x_j}\sigma_{X,i,\ell}\sigma_{X,j,\ell} \\
\sigma_C &= \nabla \pi_C\,\sigma_X .
\end{aligned}$$

Example: Black-Scholes Model

- Two assets: money market account M and stock S

$$
\begin{aligned}
\mathrm{d}S_t &= S_t[\mu \mathrm{d}t + \sigma \mathrm{d}W_t] \\
\mathrm{d}M_t &= M_t r \mathrm{d}t
\end{aligned}
$$

- This can be written in standard state space form by letting the state variable = asset prices be of dimension $n = m = 2$, with components S_t and M_t.
- There is only one source of uncertainty ($k = 1$).
- The vector functions μ_X, σ_X, and π_Y are given by

$$
\mu_X(t, S_t, M_t) = \begin{bmatrix} \mu S_t \\ r M_t \end{bmatrix}, \quad \sigma_X(t, S_t, M_t) = \begin{bmatrix} \sigma S_t \\ 0 \end{bmatrix},
$$

$$
\pi_Y(t, S_t, M_t) = \begin{bmatrix} S_t \\ M_t \end{bmatrix}.
$$

- A *Vasicek process* or *Ornstein-Uhlenbeck process* is a process of the form

$$\mathrm{d}X_t = a(b - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t.$$

- Properties: X_t fluctuates around the mean-reversion level b. The parameter a determines the mean-reversion speed. We will see later on that this process is normally distributed.
- Vasicek processes are commonly used to model rates such as interest rates, inflation rates, exchange rates, (expected) growth rates, etc.
- The Vasicek process has the (dis-)advantage that it can take positive *and* negative values.
- A prominent alternative is the *Cox-Ingersoll-Ross process*

$$\mathrm{d}X_t = a(b - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}W_t,$$

which can only take positive values, but has a very complicated distribution (non-central χ^2).

Stochastic Interest Rates: Vasicek / CIR Model

Model with Stochastic Interest Rates

- The short rate follows a Vasicek process:

$$
\begin{aligned}
\mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}W_{1,t} \\
\mathrm{d}M_t &= r_t M_t \,\mathrm{d}t \\
\mathrm{d}r_t &= a(b - r_t)\,\mathrm{d}t + \sigma_r \,\mathrm{d}\left(\rho W_{1,t} + \sqrt{1-\rho^2}\, W_{2,t}\right).
\end{aligned}
$$

- $n = 3$ state variables, S_t, M_t, r_t, along with $k = 2$ sources of risk, and $m = 2$ assets S_t, M_t. Vector/matrix functions:

$$
\mu_X(t, S_t, M_t, r_t) = \begin{bmatrix} \mu S_t \\ r_t M_t \\ a(b - r_t) \end{bmatrix},
$$

$$
\sigma_X(t, S_t, M_t, r_t) = \begin{bmatrix} \sigma_S S_t & 0 \\ 0 & 0 \\ \sigma_r \rho & \sigma_r \sqrt{1-\rho^2} \end{bmatrix}, \quad \pi_Y(t, S_t, M_t, r_t) = \begin{bmatrix} S_t \\ M_t \end{bmatrix}.
$$

- If the asset i has a positive price, i.e., π_C maps to the positive real numbers, we can rewrite

$$\begin{aligned} \mathrm{d}C_t &= \mu_C(t, X_t)\,\mathrm{d}t + \sigma_C(t, X_t)\,\mathrm{d}W_t \\ &= C_t\left[\widetilde{\mu}_C(t, X_t)\,\mathrm{d}t + \widetilde{\sigma}_C(t, X_t)\,\mathrm{d}W_t\right] \end{aligned}$$

  with $\widetilde{\mu}_C = \frac{\mu_C}{C}$, $\widetilde{\sigma}_C = \frac{\sigma_C}{C}$.
- Applying Itô's lemma to determine log return:

$$\begin{aligned} \mathrm{d}\log(C) &= C^{-1}\mathrm{d}C + \tfrac{1}{2}(-C^{-2})\mathrm{d}[C] \\ &= \widetilde{\mu}_C\,\mathrm{d}t + \widetilde{\sigma}_C\,\mathrm{d}W_t - \tfrac{1}{2}\widetilde{\sigma}_C\widetilde{\sigma}_C'\,\mathrm{d}t \end{aligned}$$

- Consequently,

$$\log(C_t) = \log(C_0) + \int_0^t (\widetilde{\mu}_C - \tfrac{1}{2}\widetilde{\sigma}_C\widetilde{\sigma}_C')\mathrm{d}s + \int_0^t \widetilde{\sigma}_C\,\mathrm{d}W_s$$

$$\implies \quad C_t = C_0 \exp\left(\int_0^t (\widetilde{\mu}_C - \tfrac{1}{2}\widetilde{\sigma}_C\widetilde{\sigma}_C')\mathrm{d}s + \int_0^t \widetilde{\sigma}_C\,\mathrm{d}W_s\right) > 0$$

Self-financing Portfolios

- ϕ_t is the vector of number of units of assets held at time t.
- Portfolio value generated by the *portfolio strategy* ϕ:

$$V_t = \phi_t' Y_t.$$

- A portfolio strategy ϕ is *self-financing* if portfolio rebalancing neither generates nor destroys money, i.e.,

$$\mathrm{d}V_t = \phi_t' \,\mathrm{d}Y_t$$

or equivalently, $V_T = V_0 + \int_0^T \phi_t' \,\mathrm{d}Y_t$. This is the self-financing condition for continuous trading.

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Checking if a Market is Free of Arbitrage

- We consider our generic state space market model

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\ Y_t &= \pi_Y(t, X_t). \end{aligned}$$

- A natural question is whether there is an easy-to-check criterion on whether a market satisfies "nice" economic properties.
- Two fundamental economic properties are
  - absence of arbitrage ("no free profits without risk")
  - completeness ("all risks are hedgeble")
- Since the model is formulated in terms of the functions $\mu_X(t, X_t)$, $\sigma_X(t, X_t)$, and $\pi_Y(t, X_t)$, it should be possible to relate these conditions to these functions.

Arbitrage Opportunity

Definition (Arbitrage Opportunity)

1. A self-financing trading strategy ϕ is said to be an arbitrage opportunity if the value V generated by ϕ satisfies the following conditions:

Arb 1.)	$V_0 = 0$	Zero net investment
Arb 2.)	$\mathbb{P}(V_T \geq 0) = 1$	Riskless investment
Arb 3.)	$\mathbb{P}(V_T > 0) > 0$	Chance of making profits

2. A market model is called free of arbitrage if no arbitrage opportunities exist.

"An arbitrage opportunity makes something out of nothing."

Working with a Numéraire

- Asset prices are expressed in terms of a chosen currency (euro, dollar, ...). For theoretical purposes it is often useful to work with a *numéraire*, and to consider *relative* asset price processes (i.e., relative to the numéraire).
- A numéraire N_t is any *asset* (or more generally a self-financing portfolio) whose price is always *strictly positive*, i.e., it has a representation

$$\begin{aligned} \mathrm{d}N_t &= \mu_N(t, X_t)\mathrm{d}t + \sigma_N(t, X_t)\mathrm{d}W_t \\ &= N_t[\widetilde{\mu}_N(t, X_t)\mathrm{d}t + \widetilde{\sigma}_N(t, X_t)\mathrm{d}W_t] \end{aligned}$$

- A portfolio strategy ϕ_t is self-financing if and only if $\mathrm{d}(V_t/N_t) = \phi_t' \mathrm{d}(Y_t/N_t)$. The relative value process is then given by

$$\frac{V_t}{N_t} = \frac{V_0}{N_0} + \int_0^t \phi_s' \,\mathrm{d}\left(\frac{Y_s}{N_s}\right).$$

First Fundamental Theorem of Asset Pricing

- Given: joint process of asset prices $(Y_t)_{t\geq 0}$, and a numéraire $(N_t)_{t\geq 0}$.

First Fundamental Theorem of Asset Pricing

The following are equivalent:

1. The market is free of arbitrage.
2. There is a probability measure $\mathbb{Q}_N \sim \mathbb{P}$ such that $(Y_t/N_t)_{t\geq 0}$ is a martingale under $\mathbb{Q}_N$.

- The measure $\mathbb{Q}_N$ is called an *equivalent martingale measure* (EMM) that corresponds to the numéraire N.
- The direction (2) $\Longrightarrow$ (1) can be proven easily. However, it is a hard task to prove (1) $\Longrightarrow$ (2), because one has to construct an EMM (see Delbean and Schachermayer 2006, *The Mathematics of Arbitrage*) .

Proof of the Easy Part

Proof of the Easy Part (cont'd)

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Proposition (No Arbitrage Criterion)

The generic state space model

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t, \qquad & Y_t = \pi_Y(t, X_t), \\ \mathrm{d}Y_t &= \mu_Y(t, X_t)\,\mathrm{d}t + \sigma_Y(t, X_t)\,\mathrm{d}W_t \end{aligned}$$

is free of arbitrage if and only if for all t and x there exists a scalar $r(t, x) \in \mathbb{R}$ and a vector $\lambda(t, x) \in \mathbb{R}^k$ such that

$$\mu_Y(t, x) - r(t, x)\pi_Y(t, x) = \sigma_Y(t, x)\lambda(t, x).$$

Another way to write the equation above:

$$\underbrace{\begin{bmatrix} \sigma_Y & \pi_Y \end{bmatrix}}_{\in \mathbb{R}^{m \times (k+1)}} \underbrace{\begin{bmatrix} \lambda \\ r \end{bmatrix}}_{\in \mathbb{R}^{k+1}} = \underbrace{\mu_Y}_{\in \mathbb{R}^m}$$

Typical Situations

- A *sufficient* condition for absence of arbitrage is that the matrix $[\sigma_Y(t,x) \quad \pi_Y(t,x)]$ is invertible for all t and x. Under this condition, the solution is moreover unique.
- The size of the matrix $[\sigma_Y(t,x) \quad \pi_Y(t,x)]$ is $m \times (k+1)$, where m is the number of assets and k is the number of Brownian motions in the model. So, for the matrix to be invertible, we need

$$m = k + 1$$

  (the number of assets exceeds the number of risk factors by one).
- If $m < k+1$, typically absence of arbitrage holds, but the solution is not unique. If $m > k+1$, then special conditions must be satisfied to prevent arbitrage.

Money Market Account I

- Notice that on every arbitrage-free market, there exists a short-term interest rate $r_t = r(t, X_t)$ (*short rate*).

- The natural numéraire (the one that is used if there is no specific reason to choose another one) is the *money market account* which is *locally risk-free* and defined by

$$\mathrm{d}M_t = r_t M_t \, \mathrm{d}t$$

- The money market account evolves according to

$$M_t = M_0 \exp\left(\int_0^t r_s \, \mathrm{d}s \right)$$

- Oftentimes, M is already specified in the dynamics of Y.

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- If the market is free of arbitrage, but M is *not* a component of Y, one can equip the market with a money market account by enlarging the price vector $\widetilde{\pi}_Y = [\pi_Y \;\; M]'$.
- The extended market is free of arbitrage and pins down the term r in the NA criterion. The following equation is trivially satisfied:

$$\begin{bmatrix} \sigma_M & \pi_M \end{bmatrix} \begin{bmatrix} \lambda \\ r \end{bmatrix} = \mu_M$$

- If the solution for r is unique (but not necessarily the solution for λ), one can indeed construct the money market account, i.e., construct a self-financing portfolio s.t. $\phi' Y = M$.
- **Moral**: Every arbitrage-free market can be equipped with an MMA such that the extended market is still free of arbitrage. Thus, the MMA can be used as a numéraire in any arbitrage-free market.

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- The process $\lambda_t = \lambda(t, X_t)$ is called the *market price of risk*.
- Given the market price of risk, we can apply Girsanov's theorem and define the Girsanov kernel

$$\theta_t = \mathcal{E}(\lambda)_t = \exp\left(-\int_0^t \lambda_s' \mathrm{d}W_s - \frac{1}{2}\int_0^t \|\lambda_s\|^2 \mathrm{d}s \right)$$

- Then the process $W^{\mathbb{Q}}$ with

$$\mathrm{d}W_t^{\mathbb{Q}} = \lambda_t \,\mathrm{d}t + \mathrm{d}W_t$$

  is a k-dimensional Brownian motion under $\mathbb{Q} \sim \mathbb{P}$.
- **Remark:** This measure $\mathbb{Q} = \mathbb{Q}_M$ is an equivalent martingale measure corresponding to the money market account as numéraire (see slide 72), a so-called *risk-neutral probability measure*.
- **Remark:** Under $\mathbb{Q}$ every traded asset has a drift rate of $r_t = r(t, X_t)$

- The condition for absence of arbitrage in the generic state space model can be written briefly as: there must exist $r = r(t, x)$ and $\lambda = \lambda(t, x)$ such that

$$\mu_Y - r\pi_Y = \sigma_Y \lambda.$$

- We will derive this from the *First Fundamental Theorem of Asset Pricing*. The following concepts will be used:
    - numéraire
    - money market account
    - equivalent martingale measure (EMM)

Proof of the NA Criterion

- Let $\mathbb{Q}_N$ denote a probability measure defined by the RN process λ_N. $\mathbb{Q}_N$ is an EMM if and only if the relative asset price process Y_t/N_t is a $\mathbb{Q}_N$-martingale, i.e., its drift rate under $\mathbb{Q}_N$ is zero.
- The relative asset price process follows

$$\mathrm{d}(Y/N) = \mu_{Y/N}\,\mathrm{d}t + \sigma_{Y/N}\,\mathrm{d}W.$$

- According to Girsanov's Theorem

$$\mathrm{d}\widetilde{W}_t = \lambda_N(t, X_t)\,\mathrm{d}t + \mathrm{d}W_t$$

is a Brownian motion under $\mathbb{Q}_N$. Therefore,

$$\mathrm{d}(Y/N) = \mu_{Y/N}\,\mathrm{d}t + \sigma_{Y/N}\,(\mathrm{d}\widetilde{W}_t - \lambda_N \mathrm{d}t).$$

- Thus, Y/N is a $\mathbb{Q}_N$-martingale if and only if $\mu_{Y/N} = \sigma_{Y/N}\lambda_N$.

Proof of the NA Criterion (cont'd)

- Choose $N_t = M_t$ (money market account) and write $\lambda_M = \lambda$.
- From $\mathrm{d}M_t = r_t M_t \,\mathrm{d}t$ it follows that

$$\mathrm{d}(M_t^{-1}) = -r_t M_t^{-1}\,\mathrm{d}t.$$

- Therefore by the stochastic product rule,

$$\mathrm{d}(Y/M) = Y\,\mathrm{d}(M^{-1}) + M^{-1}\,\mathrm{d}Y = M^{-1}\big(\mathrm{d}Y - rY\,\mathrm{d}t\big)$$

so that

$$\mu_{Y/M} = M^{-1}(\mu_Y - r\pi_Y), \qquad \sigma_{Y/M} = M^{-1}\sigma_Y.$$

- Because M^{-1} is never zero, the condition $\mu_{Y/M} = \sigma_{Y/M}\lambda$ is equivalent to the *no-arbitrage criterion*

$$\mu_Y - r\pi_Y = \sigma_Y \lambda.$$

Example: Black-Scholes Model

- Asset dynamics

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \qquad \mathrm{d}M_t = rM_t \,\mathrm{d}t.$$

- The no-arbitrage criterion $\mu_Y - r\pi_Y = \sigma_Y \lambda$ becomes

$$\begin{bmatrix} \mu S \\ rM \end{bmatrix} - r \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} \sigma S \\ 0 \end{bmatrix} \lambda$$

  where the quantities that are to be determined are indicated in blue.
- There is a (unique) solution, i.e., the BS model is free of arbitrage (and complete):

$$r = r, \quad \lambda = \frac{\mu - r}{\sigma}$$

- The $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ is given by $W_t^{\mathbb{Q}} = \lambda t + W_t$. Hence, the dynamics under $\mathbb{Q}$ are

$$\mathrm{d}S_t = rS_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t^{\mathbb{Q}}, \qquad \mathrm{d}M_t = rM_t \,\mathrm{d}t.$$

Example: Model with Stochastic Interest Rates

- The short rate follows the Vasicek model:

$$
\begin{aligned}
\mathrm{d}M_t &= r_t M_t \,\mathrm{d}t \\
\mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}W_{1,t} \\
\mathrm{d}r_t &= a(b - r_t)\,\mathrm{d}t + \sigma_r \,\mathrm{d}\big(\rho W_{1,t} + \sqrt{1-\rho^2}\, W_{2,t}\big).
\end{aligned}
$$

- No-arbitrage criterion

$$
\begin{bmatrix} \mu S \\ rM \end{bmatrix} - r \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} \sigma_S S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}
$$

- There is a (non-unique) solution. The model is free of arbitrage.
- The solution is non-unique because λ_2 is arbitrary. The quantities r and λ_1 are defined uniquely by absence of arbitrage.

(a) For a given numéraire N, derive the dynamics of Y/N.

(b) Show how the result from (a) simplifies if one chooses $N = M$.

Solution:

Problem: Working with a Numéraire

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The Pricing Problem

- Let an arbitrage-free model be given in the generic state space form

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t, \\ \mathrm{d}Y_t &= \mu_Y(t, X_t)\,\mathrm{d}t + \sigma_Y(t, X_t)\,\mathrm{d}W_t, \qquad Y_t = \pi_Y(t, X_t) \end{aligned}$$

s.t.

$$\mu_Y(t, X_t) - r(t, X_t)\pi_Y(t, X_t) = \sigma_Y(t, X_t)\lambda(t, X_t).$$

- Suppose now that a new asset is introduced, for instance a contract that will produce a state-dependent payoff at a given time $T > 0$. Pricing on the basis of absence of arbitrage means: the new asset should be priced such that no arbitrage is introduced.
- We want to turn this principle into a *pricing formula*.

Pricing Formula

- If there is an EMM $\mathbb{Q}_N$, for a given numéraire N_t, the relative price of *any* asset must be a martingale under $\mathbb{Q}_N$. By the martingale property, we therefore have:

Numéraire-dependent pricing formula

Let C_T denote the terminal payoff of a contingent claim that matures at time T. For every EMM $\mathbb{Q}_N$ for a given numéraire N_t, an arbitrage-free price at time t is given by

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right].$$

- This can be used as a *pricing formula* for derivative contracts.
- **Crucial question**: When is the arbitrage-free price of the derivative unique?

- To have uniquely defined prices of derivatives, the equation

$$\mu_Y(t,x) - r(t,x)\pi_Y(t,x) = \sigma_Y(t,x)\lambda(t,x)$$

needs to have a *unique* solution $\big(r(t,x), \lambda(t,x)\big)$. Then the corresponding EMM and the corresponding SDF are uniquely determined.

- One can show that the solution is unique if and only if the matrix $[\pi_Y \quad \sigma_Y]$ has full column rank for all (t,x).
  - Sufficient condition: the matrix $[\pi_Y \quad \sigma_Y]$ is invertible (requires $m = k+1$).
  - Necessary condition: $m \geq k+1$
- In arbitrage-free markets with unique EMM $\mathbb{Q}_N$, the arbitrage-free price $C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]$ is uniquely determined.
- We will see later on that uniqueness of the EMM corresponds to an important economic property: *market completeness*.

Verification of Absence of Arbitrage

- The process C_t is defined by

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]$$

  where C_T is a given random variable.
- In applications, the terminal payoff of the derivative, C_T, is a function of the state vector at time T: $C_T = F(X_T)$.
- To ensure that no arbitrage is introduced by the price process C_t, we need to verify that the process $(C_t/N_t)_{t\geq 0}$ is a martingale; i.e., the martingale property holds for *any* s and t with $s < t$, not just for t and T.
- This follows from the tower law of conditional expectations:

$$E_s^{\mathbb{Q}_N}\left[\frac{C_t}{N_t}\right] = E_s^{\mathbb{Q}_N}\left[E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]\right] = E_s^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right] = \frac{C_s}{N_s}\,.$$

Money Market Account as a Numéraire

- In principle, every self-financing portfolio which generates positive wealth can act as a numéraire.
- However, there are several commonly used choices:
  - Money market account
  - Stock
  - Numéraire portfolio
  - ...
- Using the money market account as a numéraire, the pricing formula becomes

$$C_t = B_t E_t^{\mathbb{Q}}\left[\frac{C_T}{B_T}\right] = E_t^{\mathbb{Q}}\left[C_T \frac{B_t}{B_T}\right] = E_t^{\mathbb{Q}}\left[C_T \, \mathrm{e}^{-\int_t^T r_s ds}\right]$$

- We refer to $\mathbb{Q} = \mathbb{Q}_M$ as the *risk-neutral pricing measure*. Under $\mathbb{Q}$, the agent discounts at the risk-free rate and does not require a risk premium.
- Under $\mathbb{Q}$ every *traded asset* has an expected return of $r = r(t, X_t)$.

The Numéraire Portfolio

- **Natural question**: Is there a numéraire N for which $\mathbb{Q}_N = \mathbb{P}$?
- In an arbitrage free market driven by Brownian motion, one can show that the answer is positive if one can solve the problem of maximizing expected log-utility from terminal wealth, i.e., if

$$\max_{\phi} \mathbb{E}[\log(V_T^{\phi})] < \infty$$

- The portfolio ρ that maximizes this optimization problem will be called the *log-optimal portfolio* or the *numéraire portfolio*.
- One can show that using the numéraire portfolio as numéraire N, the pricing formula becomes the *real-world pricing formula*

$$C_t = \mathbb{E}_t\left[C_T \frac{V_t^{\rho}}{V_T^{\rho}}\right]$$

where the expectation is calculated under $\mathbb{P}$.

Alternative Formulation of the FTAP

- Instead of exploiting an eqivalent martingale measure, it is also very common to make use of a *stochastic discount factor* (SDF) or *pricing kernel*.
- A stochastic discount factor K is a positive adapted process with $K_0 = 1$ such that the process $(K_t\, Y_t)$ is a martingale under $\mathbb{P}$, i.e.,

$$\mathbb{E}_t[K_s Y_s] = K_t Y_t$$

- One can show that the existence of an EMM is equivalent to the existence of a SDF. Therefore, the FTAP can also be formulated in terms of the SDF:

First Fundamental Theorem of Asset Pricing

The following are equivalent:

1. The market is free of arbitrage.
2. There is a stochastic discount factor.

Some Properties of the SDF

- The SDF is a positive adapted process, i.e., it can be written as (see slide 46)

$$K_t = \exp\left(\int_0^t (\widetilde{\mu}_K - \tfrac{1}{2}\widetilde{\sigma}_K \widetilde{\sigma}_K')\mathrm{d}s + \int_0^t \widetilde{\sigma}_K \,\mathrm{d}W_s \right)$$

- By definition of the SDF, the process $KM = (K_t M_t)_{t \geq 0}$ must be a martingale under $\mathbb{P}$. It follows from Itô's lemma that

$$\mathrm{d}(KM)_t = K_t M_t[(r + \widetilde{\mu}_K)\mathrm{d}t + \widetilde{\sigma}_K \mathrm{d}W_t]$$

  where $\widetilde{\sigma}_K = -\lambda'$. The martingale property implies $\widetilde{\mu}_K = -r$.
- The SDF combines the role of discounting at the short rate and the change of measure from $\mathbb{P}$ to $\mathbb{Q}$.
- It follows that the numéraire portfolio and the pricing kernel are inversely related, i.e., $K_t = \frac{1}{V_t^\rho}$.

Multiple Payoffs

- A contract may generate payoffs (possible uncertain) at multiple points in time.
- Such a contract can be viewed as a portfolio of options with individual payoff dates. The value of the portfolio is the sum of the values of its constituent parts.
- We get, for a contract with payoffs $\hat{C}_{T_i}$ at times T_i $(i = 1, \dots, n)$:

$$C_0 = N_0 \sum_{i=1}^{n} E^{\mathbb{Q}_N}\left[\frac{\hat{C}_{T_i}}{N_{T_i}}\right].$$

- In the special case of constant interest rates, we can take the money market account $M_t = e^{rt}$ as the numéraire; then

$$C_0 = \sum_{i=1}^{n} e^{-rT_i} E^{\mathbb{Q}}[\hat{C}_{T_i}].$$

- This shows that the NDPF can be seen as a generalized *net present value formula*.

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Replication

- So far, we have talked about no-arbitrage and uniqueness of arbitrage-free prices. We now turn to the natural question of whether we can hedge risks and replicate payoffs.
- Let an arbitrage-free model be given in the generic state space form

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t, \\ \mathrm{d}Y_t &= \mu_Y(t, X_t)\,\mathrm{d}t + \sigma_Y(t, X_t)\,\mathrm{d}W_t, \qquad Y_t = \pi_Y(t, X_t) \end{aligned}$$

s.t.

$$\mu_Y(t, X_t) - r(t, X_t)\pi_Y(t, X_t) = \sigma_Y(t, X_t)\lambda(t, X_t).$$

- If we want to price a claim, a natural question is whether this derivative can be replicated by a self-financing trading strategy ϕ.

Replication and Complete Market

Definition (Replication Strategy, Completeness)

Let $C_T = F(X_T)$ be the terminal payoff of a contingent claim.

1. A self-financing portfolio strategy ϕ is called a *replication strategy* or *hedging strategy* for C if

$$V_T^{\phi} = C_T$$

2. The claim is said to be *attainable* if there exists a replication strategy ϕ for this claim.
3. A market is said to be complete if and only if every claim is attainable.

- A replication strategy is thus a portfolio whose value is, under all circumstances, equal to the value of a specified contingent claim.
- Market completeness is a desirable property but typically not met in reality.

Pricing by Replication

Lemma (Law of One Price)

Suppose the market is arbitrage-free.

1. For an attainable contingent claim C with hedging strategy ϕ,

$$C_0 = V_0^{\phi}$$

is the unique arbitrage-free price, i.e., trading in the primary assets *and* the derivative does not lead to arbitrage opportunities.

2. If $V_T^{\phi} = V_T^{\psi}$ for trading strategies ϕ and ψ, then

$$V_0^{\phi} = V_0^{\psi}.$$

- The proof is trivial and does not rely on specific asset dynamics.

When is Replication Possible?

- We need an easy-to-check criterion when replication is possible.

Second Fundamental Theorem of Asset Pricing

For an arbitrage-free market, the following are equivalent:

1. The market is complete.
2. For any given numéraire N, the corresponding EMM $\mathbb{Q}_N \sim \mathbb{P}$ is unique.

- We have already seen that for an arbitrage-free market, the EMM is unique if and only if the matrix $[\pi_Y(t,x)\ \sigma_Y(t,x)] \in \mathbb{R}^{m\times(k+1)}$ has full column rank for all (t,x).
- Consequently, if there are enough traded assets ($m > k+1$ is necessary) in the model to determine prices uniquely, then they are also enough to make replication possible. And vice versa.

Examples

- Obviously, the Black Scholes model (see slides 42, 63) is complete since
$$[\pi_Y \ \sigma_Y] = \begin{bmatrix} S_t & S_t\sigma \\ M_t & 0 \end{bmatrix}$$
is invertible for every combination of S_t and $M_t > 0$. Besides, there was a unique solution for r and λ, which uniquely determines the change of measure.

  $\Longrightarrow$ Pricing by replication is always possible.

- The model with stochastic interest rates of the Vasicek type (see slides 45, 64) is incomplete ($m = k = 2$), and the EMM is not unique since there is no unique solution for λ_2.

  $\Longrightarrow$ Pricing by replication is in general impossible.

  However, the model can be completed by adding a bond that can be used to hedge interest rate risk (see Chapter 6).

To replicate a payoff at time T given by $C_T = F(X_T)$, we follow a four-step procedure:

Step 1. Choose a numéraire N_t and determine the function

$$\pi_C(t,x) = \pi_N(t,x)\, E^{\mathbb{Q}_N}\Big[\, \frac{F(X_T)}{\pi_N(T,X_T)} \,\Big|\, X_t = x \Big].$$

Step 2. Compute $\sigma_C(t,x) = \nabla \pi_C(t,x)\, \sigma_X(t,x)$.

Step 3. Solve for $\phi = \phi(t,x)$ from

$$[\sigma_C \quad \pi_C] = \phi' [\sigma_Y \quad \pi_Y].$$

Step 4. Start with initial capital $\pi_C(0,X_0)$, and rebalance your portfolio along the trading strategy $\phi_t = \phi(t,X_t)$.

Validity of the replication recipe

- To show the validity of the replication recipe, three conditions need to be demonstrated:
   (i) the equation $[\sigma_C \quad \pi_C] = \phi'[\sigma_Y \quad \pi_Y]$ (where ϕ is the unknown) can be solved
   (ii) the portfolio value generated by the trading strategy ϕ at time T is equal to $V_T^{\phi} = F(X_T)$.
   (iii) the trading strategy $\phi_t = \phi(t, X_t)$ is self-financing
- These items will be discussed on the next slides.

Property of the function π_C

- We already know that the process defined by $C_t = \pi_C(t, X_t)$ with

$$\pi_C(t,x) = \pi_N(t,x)\,\mathbb{E}_t^{\mathbb{Q}_N}\left[\frac{F(X_T)}{\pi_N(T,X_T)}\right]$$

  is such that C_t/N_t is a martingale under $\mathbb{Q}_N$.
- This property is translated into state space terms as follows: let $r = r(t,x)$ and $\lambda = \lambda(t,x)$ be defined as the solution of the equation (NA criterion):

$$\mu_Y - r\pi_Y = \sigma_Y \lambda.$$

- Then we also have

$$\mu_C - r\pi_C = \sigma_C \lambda.$$

- Market completeness means that the EMM for any given numéraire is uniquely defined, i.e., the equation

$$\underbrace{\mu_Y}_{\in\mathbb{R}^m} = \underbrace{\begin{bmatrix}\sigma_Y & \pi_Y\end{bmatrix}}_{\in\mathbb{R}^{m\times(k+1)}} \underbrace{\begin{bmatrix}\lambda \\ r\end{bmatrix}}_{\in\mathbb{R}^{k+1}}$$

  has a *unique* solution $[\lambda \;\; r]'$.
- In other words, the matrix $[\sigma_Y \;\; \pi_Y] = [\sigma_Y(t,x) \;\; \pi_Y(t,x)]$ has rank $k+1$ for all t and x (its columns are linearly independent).
- Because row rank = column rank, this implies that the *rows* of the matrix span the $(k+1)$-dimensional space. This means that the equation

$$[\sigma_C \;\; \pi_C] = \phi'[\sigma_Y \;\; \pi_Y].$$

  has a unique solution ϕ. So requirement (i) is indeed satisfied.

- Define the portfolio strategy $\phi_t = \phi(t, X_t)$. The corresponding portfolio value is $V_t = \phi_t' Y_t$. Because $\phi' \pi_Y = \pi_C$, this implies that $V_t = C_t$ for all t. In particular, $V_T = F(X_T)$ (requirement (ii)).

- Because $\phi' \pi_Y = \pi_C$ and $\phi' \sigma_Y = \sigma_C$, and because $\mu_Y = r\pi_Y + \sigma_Y \lambda$ as well as $\mu_C = r\pi_C + \sigma_C \lambda$, we have

$$\phi' \mu_Y = \phi'(r\pi_Y + \sigma_Y \lambda) = r\phi' \pi_Y + \phi' \sigma_Y \lambda = r\pi_C + \sigma_C \lambda = \mu_C.$$

- Therefore,

$$\mathrm{d}V = \mu_C \,\mathrm{d}t + \sigma_C \,\mathrm{d}W = \phi'(\mu_Y \,\mathrm{d}t + \sigma_Y \,\mathrm{d}W) = \phi' \mathrm{d}Y$$

which shows that the proposed portfolio strategy is self-financing (requirement (iii)).

- BS model under $\mathbb{Q}$ (check!):

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}} \\ \mathrm{d}M_t &= rM_t\,\mathrm{d}t. \end{aligned}$$

- Payoff at time T: $\max(S_T - K, 0)$.
- Step 1: determine the pricing function:

$$\pi_C(t, S_t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$

with

$$d_{1,2} = \frac{\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Example: Call Option in BS Model (cont'd)

- Step 2: compute

$$\sigma_C(t, S_t) = \frac{\partial \pi_C}{\partial S_t}(t, S_t)\, \sigma S_t = \Phi(d_1)\sigma S_t.$$

- Step 3: solve for $\phi(t, S_t) = [\phi_S(t, S_t)\ \ \phi_M(t, S_t)]$ from

$$[\Phi(d_1)\sigma S_t \quad S_t\Phi(d_1) - e^{-r(T-t)} K\Phi(d_2)] = [\phi_S \quad \phi_M] \begin{bmatrix} \sigma S_t & S_t \\ 0 & M_t \end{bmatrix}.$$

- We find

$$\phi_S(t, S_t) = \Phi(d_1)$$

$$\phi_M(t, S_t) = -K\,\Phi(d_2)$$

Delta Hedging

- The "delta" of an option is the derivative of the option price with respect to the value of the underlying Y_i, i.e.,

$$\Delta_C = \frac{\partial \pi_C}{\partial Y_i}$$

- There could be several underlying assets (for instance in the case of an option written on the maximum of two stocks), and in that case there are also several deltas.
- In models driven by a single Brownian motion, if an option depends on a single underlying asset, then the number of units of the underlying asset that should be held in a replicating portfolio is given by the delta of the option (as in the example). The resulting strategy is called the *delta hedge*.
- Under certain conditions this also works in the case of multiple underlyings.

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- Compute μ_C and σ_C (Itô's lemma):

$$\mu_C = \frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X \sigma_X'\Big)$$

$$\sigma_C = \nabla \pi_C \sigma_X.$$

- The equation $\mu_C - r\pi_C = \sigma_C \lambda$ becomes:

Pricing PDE

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \underbrace{(\mu_X - \sigma_X \lambda)}_{=\mu_X^{\mathbb{Q}_N}} + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X \sigma_X'\Big) = r\pi_C, \qquad \pi_C(T, x) = F(x)$$

- This is a *partial differential equation* for the pricing function π_C.
- Notice that the *boundary condition* $\pi_C(T, x) = F(x)$ determines the type of the derivative.

Remarks

- In a model without any non-traded state variables, i.e., $Y = X$, $\pi_Y = x$, the NA condition becomes

$$\mu_X - \sigma_X \lambda = r\,x$$

- The PDE collapses to

$$\frac{\partial \pi_C}{\partial t} + r \nabla \pi_C \cdot x + \frac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X \sigma_X'\Big) = r\pi_C$$

- The drift term of the spatial first-order derivatives is r, which is the drift term of traded assets under $\mathbb{Q}$.
- The PDE may be solved analytically or numerically (finite-difference methods – generalization of tree methods).
- The PDE can also be derived using the Feynman-Kac Theorem: a mathematical statement that connects the theory of partial differential equations to conditional expectations.

Excursion: The Feynman-Kac Theorem

Theorem (Feynman-Kac)

Consider the following parabolic partial differential equation

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X^{\mathbb{Q}}(t,x) + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X(t,x)\sigma_X(t,x)'\Big) + f(t,x) = r(t,x)\pi_C$$

subject to the terminal condition $\pi_C(T,x) = F(x)$. Then, the solution can be written as a conditional expectation

$$\pi_C(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[\int_t^T e^{-\int_t^s r(\tau,X_\tau)\,d\tau} f(s,X_s)\mathrm{d}s + e^{-\int_t^T r(\tau,X_\tau)\,d\tau} F(X_T)\right]$$

under $\mathbb{Q}$ such that X is an Itô process driven by the equation

$$\mathrm{d}X = \mu_X^{\mathbb{Q}}(t,X)\,\mathrm{d}t + \sigma_X(t,X)\,\mathrm{d}W^{\mathbb{Q}},$$

with $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

Example: Black-Scholes PDE

- Under $\mathbb{Q}$, the dynamics are

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}} \\ \mathrm{d}M_t &= rM_t\,\mathrm{d}t. \end{aligned}$$

- Therefore, the BSPDE for a derivative with terminal payoff $F(S_T)$ reads

$$\frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} Sr + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2} S^2\sigma^2 = r\,\pi_C$$

  s.t. $\pi_C(T, S_T) = F(S_T)$
- In their original paper Black and Scholes (1973), derived this formula using a different approach and made two mistakes which cancel each other out. Merton (1973) corrected these mistakes and came up with the same PDE.
- The PDE can be transformed to the so-called *heat equation*, which is commonly used in physics and has a well-known solution.

Example: Pricing PDE with Stoch. Interest Rates

- Under $\mathbb{Q}$, the dynamics are

$$
\begin{aligned}
\mathrm{d}M_t &= r_t M_t \,\mathrm{d}t \\
\mathrm{d}S_t &= r_t S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}W_{1,t}^{\mathbb{Q}} \\
\mathrm{d}r_t &= a^{\mathbb{Q}}(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma_r \,\mathrm{d}\big(\rho W_{1,t}^{\mathbb{Q}} + \sqrt{1-\rho^2}\, W_{2,t}^{\mathbb{Q}}\big).
\end{aligned}
$$

- Notice that the risk-neutral measure is not uniquely determined since the market price of risk $\lambda = (\lambda_1 \ \lambda_2)$ is not unique.
- Therefore, the pricing PDE for a derivative with payoff $F(r_T, S_T)$ reads

$$
\begin{aligned}
r\,\pi_C = \frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} Sr + \frac{\partial \pi_C}{\partial r} a^{\mathbb{Q}}(b^{\mathbb{Q}} - r) \\
+ \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2} S^2 \sigma_S^2 + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial r^2}\sigma_r^2 + \frac{\partial^2 \pi_C}{\partial r \partial S}\rho\sigma_r\sigma_S S
\end{aligned}
$$

s.t. $\pi_C(T, r_T, S_T) = F(r_T, S_T)$

Summary

- Generic State Space Model:

$$dX_t = \mu_X(t, X_t)\,dt + \sigma_X(t, X_t)\,dW_t, \qquad Y_t = \pi_Y(t, X_t)$$

- No-arbitrage condition (from FTAP 1):

$$\mu_Y - r\pi_Y = \sigma_Y \lambda$$

- Numéraire-dependent pricing formula:

$$\frac{C_t}{N_t} = E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]$$

- Replication recipe (from FTAP 2) if $\mathrm{rk}(\sigma_Y \;\; \pi_Y) = k+1$:

$$[\sigma_C \quad \pi_C] = \phi'[\sigma_Y \quad \pi_Y]$$

- Pricing via PDE:

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot (\mu_X - \sigma_X \lambda) + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\, \sigma_X \sigma_X'\Big) = r\pi_C$$