Valuation and Risk Management 2023
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Problem Set 2

Problem 1 (Fundamental Notions)

(a) Explain why neither the Vasicek model nor the Cox-Ingersol-Ross model is able to correctly model volatilities of and correlations between short-term interest rates and long-term interest rates.

(b) State the definition of an *affine term structure model* and explain why this class is very common in quantitative finance.

(c) Explain what is understood by *interest rate risk*. How is interest rate risk different from credit risk?

(d) Explain why the Nelson-Siegel model is not free of arbitrage. Is this a big problem from a practical point of view? Which steps need to be carried out to make the model arbitrage-free?

Problem 2 (Relation between Vasicek and CIR) Consider the process

$$\mathrm{d}Y_t = (2aY_t + \sigma^2)\mathrm{d}t + 2\sigma\sqrt{Y_t}\mathrm{d}W_t$$

(a) Determine the dynamics and the distribution of the process $X_t = \sqrt{Y_t}$.

(b) What can you say about the distribution of Y_t given your calculations from part (a)?

Problem 3 (CIR Model) Consider the Cox-Ingersol-Ross Model

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma\sqrt{r_t}\,\mathrm{d}W_t^{\mathbb{Q}}$$

where r_t is the short rate, and the process $W_t^{\mathbb{Q}}$ is a Brownian motion under the risk-neutral probability measure.

(a) Determine the expectation of the future short rate $\mathbb{E}^{\mathbb{Q}}[r_T]$.

(b) It is well-known that the CIR model admits an exponentially affine representation of the price of a zero coupon bond, i.e.,

$$P_t(T) = \exp\big(A(t,T) + B(t,T)r_t\big)$$

for some functions A and B. Derive an expression for the forward rate $F_t(T_1, T_2)$ in terms of the functions A and B and the current short rate r_t. Also derive the instantaneous forward rate $F_t(T)$.

Problem 4 (Interest Rate Options) A company is planning to take out a loan at a time T_1 in the future. The loan will be paid back at time $T_2 > T_1$. At that time the company will also pay the interest on the loan. The interest rate is determined at time T_1. In this situation, the company faces the risk that at time T_1 interest rates will be high. In order to reduce this risk, the company can enter an "interest rate cap" at level $r_{\max}$. Such a contract has the following effect:

(i) when the interest rate that holds at time T_1 for loans that mature at time T_2 is *higher* than $r_{\max}$, the effective interest rate paid by the company is only $r_{\max}$;

(ii) when the interest rate that holds at time T_1 for loans that mature at time T_2 is *less* than or equal to $r_{\max}$, the rate paid by the company is the actual rate.

Continuous compounding is assumed throughout.

(a) Show that the company can achieve the effect of the interest rate cap by buying a put option on the value at time T_1 of a bond that matures at time T_2. The payoff of the put option at time T_1 is

$$\max(K - P_{T_1}(T_2), 0)B$$

where $P_t(T)$ is the value at time t of a default-free bond that pays one unit of currency at time $T \geq t$, K is the strike of the option, and B is a number that determines the size of the option contract. Given the cap level $r_{\max}$ and the amount A that the company wants to borrow at time T_1, determine the strike level K and the number B so that the put option has the desired effect.

(b) Consider now the put option of size 1, i.e., the put option with payoff $\max(K - P_{T_1}(T_2), 0)$, and assume we are working in a complete market. Using the numéraire-dependent pricing formula, show that the value of the put option at time $t < T_1$ is given by

$$P_t(T_1)K\mathbb{E}_t^{\mathbb{Q}_{T_1}}\left[1_{\{P_{T_1}(T_2)<K\}}\right] - P_t(T_2)\mathbb{E}_t^{\mathbb{Q}_{T_2}}\left[1_{\{P_{T_1}(T_2)<K\}}\right]$$

where $\mathbb{Q}_T$ denotes, for any given T, the T-terminal measure, that is, the equivalent martingale measure that corresponds to taking as a numéraire the bond that pays one unit of currency at time T.

Problem 5 (T-Forward Measure) It follows from the previous problem that the value of an interest rate cap can be determined within a given term structure model if it is possible, under any given terminal measure, to compute the probability that the bond price for a given maturity at a given future time will exceed a given level. Suppose now that we work with the Vasicek model, given in the following form:

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}$$

where r_t is the short rate, and the process $\{W_t^{\mathbb{Q}}\}$ is a Brownian motion under the risk-neutral measure. Recall that the price at time t of a bond maturing at time $T \geq t$ is given in the Vasicek model by an expression of the form

$$P_t(T) = \exp\left(A(t,T) + B(t,T)r_t)\right)$$

where A and B are deterministic functions of time. You may express your answers to the questions below in terms of f and g. Recall also the change-of-numéraire formula

$$\mathrm{d}W_t^N = \mathrm{d}W_t^{\mathbb{Q}} - \frac{\sigma_N}{\pi_N}\,\mathrm{d}t$$

where the process $\{W_t^N\}$ is a Brownian motion under the equivalent martingale measure corresponding to a new numéraire N_t.

(a) Show that, for any given T, the Vasicek model can be written in the form

$$dr_t = (-ar_t + h(t))\,dt + \sigma\,dW_t^T$$

where $h(t)$ is a deterministic function of time (which may depend on T) and the process $\{W_t^T\}$ is a Brownian motion under the T-terminal measure $\mathbb{Q}_T$. Determine the function $h(t)$ when T is given.

(b) Show that, for any given $t \leq T_1 \leq T_2$, the conditional distribution (given information up to time t) of the short rate at time T_1 in the Vasicek model, under the T_2-terminal measure, is normal. How does this help to compute the value of an interest rate cap?

Problem 6 (Credit Risk) Suppose Merton's firm value model is used to assess credit risk. The firm value evolves according to

$$\mathrm{d}V_t = V_t\big[\mu \mathrm{d}t + \sigma \mathrm{d}W_t\big],$$

where all parameters are constant, and W_t is a standard Brownian motion under the physical measure $\mathbb{P}$. The risk-free term structure is flat with yield r, and the firm has emitted a single zero-coupon bond with notional F and maturity T.

(a) Find an explicit expression for the *distance-to-default* that is the distance between the expected value of the asset and the default point.

(b) In the lecture we discussed the formula

$$\sigma \frac{\Phi\big(d_1(\sigma)\big)}{E(\sigma)} = \frac{\sigma_E}{V},$$

which relates the firm value volatility to the volatility of equity. Derive this formula.