**Valuation and Risk Management 2023**
**Tilburg School of Economics and Management**
**Christoph Hambel**
**Problem Set 2**

## Problem 1 (Fundamental Notions)

(a) Explain why neither the Vasicek model nor the Cox-Ingersol-Ross model is able to correctly model volatilities of and correlations between short-term interest rates and long-term interest rates.

**Solution:** In both models, there is only one risk factor that affects short rates and long rates in the same manner. Hence, they are perfectly correlated, which is not in line with empirical evidence.

(b) State the definition of an *affine term structure model* and explain why this class is very common in quantitative finance.

**Solution:** A term structure model with state variables $X$ is called affine if it generates $T$-bond prices of the form

$$P_t(T) = \mathrm{e}^{A(t,T)+B(t,T)'X_t}$$

for appropriate functions $A$ and $B$ satisfying $A(T,T) = 0$, $B(T,T) = 0$. Affine term structure models are quite tractable since they allow for closed-form representations of bond prices and the term structure of interest rates, which is affine in $X$ as well. Often, these models admit closed-form solutions for bond options.

(c) Explain what is understood by *interest rate risk*. How is interest rate risk different from credit risk?

**Solution:** Interest rate risk is the potential that a change in overall interest rates will reduce the value of a bond or other fixed-rate investment: As interest rates rise bond prices fall, and vice versa. While interest rate risk is a form of market risk, credit risk refers to the risk that a party fails to make a payment, e.g., that a bond issuer cannot pay her debt back. Credit risk is thus issuer-specific.

(d) Explain why the Nelson-Siegel model is not free of arbitrage. Is this a big problem from a practical point of view? Which steps need to be carried out to make the model arbitrage-free?

**Problem 2 (Relation between Vasicek and CIR)** Consider the process

$$dY_t = (2aY_t + \sigma^2)dt + 2\sigma\sqrt{Y_t}dW_t$$

(a) Determine the dynamics and the distribution of the process $X_t = \sqrt{Y_t}$.

**Solution:** We apply Itô's lemma to determine the dynamics of $X_t$ with $f(y) = \sqrt{y} = y^{1/2}$. Therefore, $f'(y) = \frac{1}{2}y^{-1/2}$, $f''(y) = -\frac{1}{4}y^{-3/2}$, and hence

$$\begin{aligned} dX_t &= f'(Y_t)dY_t + \frac{1}{2}f''(Y_t)d[Y]_t \\ &= \tfrac{1}{2}Y_t^{-1/2}(2aY_t + \sigma^2)dt + \tfrac{1}{2}Y_t^{-1/2}2\sigma\sqrt{Y_t}dW_t - \tfrac{1}{2}\cdot\tfrac{1}{4}Y_t^{-3/2}(2\sigma)^2Y_tdt \\ &= \sqrt{Y_t}adt + \sigma dW_t \\ &= aX_tdt + \sigma dW_t \end{aligned}$$

Consequently, $X$ is an Ornstein-Uhlenbeck process with mean-reversion level 0.

(b) What can you say about the distribution of $Y_t$ given your calculations from part (a)?

**Solution:** Since $Y_t = X_t^2$ and $X_t$ is normally distributed, $Y_t$ has a non-central $\chi^2$-distribution.

**Problem 3 (CIR Model)** Consider the Cox-Ingersol-Ross Model

$$dr_t = a(b - r_t)\,dt + \sigma\sqrt{r_t}\,dW_t^{\mathbb{Q}}$$

where $r_t$ is the short rate, and the process $W_t^{\mathbb{Q}}$ is a Brownian motion under the risk-neutral probability measure.

(a) Determine the expectation of the future short rate $\mathbb{E}^{\mathbb{Q}}[r_T]$.

**Solution:** This problem can be solved along the lines of slides 42 and 43. Doing exactly the same steps yields

$$\mathbb{E}^{\mathbb{Q}}[r_T] = e^{-aT}r_0 + b(1 - e^{-aT}) + \sigma\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-as}\sqrt{r_s}dW_s\Big]$$

The open question is whether $\mathbb{E}^{\mathbb{Q}}\big[\int_0^T e^{-as}\sqrt{r_s}dW_s\big] = 0$.[1] One can show (using some technical arguments, which are beyond the scope of the lecture and not relevant for examination) that $\int_0^T e^{-as}\sqrt{r_s}dW_s$ is indeed a martingale. This implies $\mathbb{E}^{\mathbb{Q}}\big[\int_0^T e^{-as}\sqrt{r_s}dW_s\big] = 0$.

[1] For the Vasicek process, this was trivial since then the integral reads $\int_0^T e^{-as}dW_s$, which is a martingale as the integrand is deterministic. Hence $\mathbb{E}^{\mathbb{Q}}\big[\int_0^T e^{-as}dW_s\big] = 0$

Thus, the expected short rate in the CIR model is identical to the expected short rate in the Vasicek model.

(b) It is well-known that the CIR model admits an exponentially affine representation of the price of a zero coupon bond, i.e.,

$$P_t(T) = \exp\big(A(t,T) + B(t,T)r_t\big)$$

for some functions $A$ and $B$. Derive an expression for the forward rate $F_t(T_1, T_2)$ in terms of the functions $A$ and $B$ and the current short rate $r_t$. Also derive the instantaneous forward rate $F_t(T)$.

**Solution:** Prices of $T$-bonds and forward rates are related as follows:

$$P_t(T_2)e^{F_t(T_1,T_2)(T_2-T_1)} = P_t(T_1)$$

or equivalently,

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \log \frac{P_t(T_1)}{P_t(T_2)}$$

Therefore,

$$\begin{aligned}
F_t(T_1, T_2) &= \frac{1}{T_2 - T_1} \log \left[\frac{\exp\big(A(t,T_1) + B(t,T_1)r_t\big)}{\exp\big(A(t,T_2) + B(t,T_2)r_t\big)}\right] \\
&= \frac{1}{T_2 - T_1}\Big[A(t,T_1) + B(t,T_1)r_t - A(t,T_2) - B(t,T_2)r_t)\Big] \\
&= \frac{1}{T_2 - T_1}\Big[A(t,T_1) - A(t,T_2) + [B(t,T_1) - B(t,T_2)]r_t\Big]
\end{aligned}$$

Consequently,

$$\begin{aligned}
F_t(T) &= \lim_{\Delta t \to 0} F_t(T, T + \Delta t) \\
&= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big[A(t,T) - A(t,T+\Delta t) + [B(t,T) - B(t,T+\Delta t)]r_t\Big] \\
&= -\frac{\partial}{\partial T}A(t,T) - \frac{\partial}{\partial T}B(t,T)r_t.
\end{aligned}$$

**Problem 4 (Interest Rate Options)** A company is planning to take out a loan at a time $T_1$ in the future. The loan will be paid back at time $T_2 > T_1$. At that time the company will also pay the interest on the loan. The interest rate is determined at time $T_1$.

In this situation, the company faces the risk that at time $T_1$ interest rates will be high. In order to reduce this risk, the company can enter an "interest rate cap" at level $r_{\max}$. Such a contract has the following effect:

(i) when the interest rate that holds at time $T_1$ for loans that mature at time $T_2$ is *higher* than $r_{\max}$, the effective interest rate paid by the company is only $r_{\max}$;

(ii) when the interest rate that holds at time $T_1$ for loans that mature at time $T_2$ is *less* than or equal to $r_{\max}$, the rate paid by the company is the actual rate.

Continuous compounding is assumed throughout.

(a) Show that the company can achieve the effect of the interest rate cap by buying a put option on the value at time $T_1$ of a bond that matures at time $T_2$. The payoff of the put option at time $T_1$ is

$$\max(K - P_{T_1}(T_2), 0)B$$

where $P_t(T)$ is the value at time $t$ of a default-free bond that pays one unit of currency at time $T \geq t$, $K$ is the strike of the option, and $B$ is a number that determines the size of the option contract. Given the cap level $r_{\max}$ and the amount $A$ that the company wants to borrow at time $T_1$, determine the strike level $K$ and the number $B$ so that the put option has the desired effect.

**Solution:** The company wants to pay at most the amount $e^{r_{\max}(T_2-T_1)}A$ at time $T_2$. The market value of this payment at time $T_1$ is $P_{T_1}(T_2)e^{r_{\max}(T_2-T_1)}A$. The option payoff at time $T_1$ should be such that if this value is less than $A$, the payoff exactly makes up for the difference. In other words, if the option payoff is denoted by $C_{T_1}$, we should have

$$\begin{aligned} C_{T_1} &= \max\left(A - P_{T_1}(T_2)e^{r_{\max}(T_2-T_1)}A,\, 0\right) = \\ &= \left(\max(e^{-r_{\max}(T_2-T_1)} - P_{T_1}(T_2),\, 0)\right) e^{r_{\max}(T_2-T_1)}A = \\ &= \left(\max(K - P_{T_1}(T_2), 0)\right) B \end{aligned}$$

with

$$K = e^{-r_{\max}(T_2-T_1)}, \qquad B = e^{r_{\max}(T_2-T_1)}A.$$

(b) Consider now the put option of size 1, i.e., the put option with payoff $\max(K - P_{T_1}(T_2), 0)$, and assume we are working in a complete market. Using the numéraire-dependent pricing formula, show that the value of the put option at time $t < T_1$ is given by

$$P_t(T_1) K \mathbb{E}_t^{\mathbb{Q}_{T_1}} \left[ 1_{\{P_{T_1}(T_2) < K\}} \right] - P_t(T_2) \mathbb{E}_t^{\mathbb{Q}_{T_2}} \left[ 1_{\{P_{T_1}(T_2) < K\}} \right]$$

where $\mathbb{Q}_T$ denotes, for any given $T$, the $T$-terminal measure, that is, the equivalent martingale measure that corresponds to taking as a numéraire the bond that pays one unit of currency at time $T$.

**Solution:** We can write

$$\max(K - P_{T_1}(T_2), 0) = 1_{P_{T_1}(T_2) < K}(K - P_{T_1}(T_2)).$$

Let the values of time $t < T_1$ of the contracts with payoffs $1_{P_{T_1}(T_2) < K} K$ and $1_{P_{T_1}(T_2) < K} P_{T_1}(T_2)$ at time $T_1$ be denoted by $C_t^1$ and $C_t^2$ respectively; then the value of the bond put at time $t$ is $C_t = C_t^1 - C_t^2$. Apply the numéraire-dependent pricing formula with numéraire $P_t(T_1)$ to $C_t^1$:

$$\frac{C_t^1}{P_t(T_1)} = E_t^{\mathbb{Q}_{T_1}} \frac{C_{T_1}^1}{P_{T_1}(T_1)} = K E_t^{\mathbb{Q}_{T_1}} 1_{P_{T_1}(T_2) < K}.$$

Apply the NDPF with numéraire $P_t(T_2)$ to $C_t^2$:

$$\frac{C_t^2}{P_t(T_2)} = E_t^{\mathbb{Q}_{T_2}} \frac{C_{T_1}^2}{P_{T_1}(T_2)} = E_t^{\mathbb{Q}_{T_2}} 1_{P_{T_1}(T_2) < K}.$$

Therefore, we have

$$C_t = C_t^1 - C_t^2 = P_t(T_1) K E_t^{\mathbb{Q}_{T_1}} 1_{P_{T_1}(T_2) < K} - P_t(T_2) E_t^{\mathbb{Q}_{T_2}} 1_{P_{T_1}(T_2) < K}.$$

**Problem 5 ($T$-Forward Measure)** It follows from the previous problem that the value of an interest rate cap can be determined within a given term structure model if it is possible, under any given terminal measure, to compute the probability that the bond price for a given maturity at a given future time will exceed a given level. Suppose now that we work with the Vasicek model, given in the following form:

$$\mathrm{d}r_t = a(b - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}$$

where $r_t$ is the short rate, and the process $\{W_t^{\mathbb{Q}}\}$ is a Brownian motion under the risk-neutral measure. Recall that the price at time $t$ of a bond maturing at time $T \geq t$ is given in the Vasicek model by an expression of the form

$$P_t(T) = \exp\big(A(t,T) + B(t,T)r_t)\big)$$

where $A$ and $B$ are deterministic functions of time. You may express your answers to the questions below in terms of $f$ and $g$. Recall also the change-of-numéraire formula

$$\mathrm{d}W_t^N = \mathrm{d}W_t^{\mathbb{Q}} - \frac{\sigma_N}{\pi_N}\,\mathrm{d}t$$

where the process $\{W_t^N\}$ is a Brownian motion under the equivalent martingale measure corresponding to a new numéraire $N_t$.

(a) Show that, for any given $T$, the Vasicek model can be written in the form

$$dr_t = (-ar_t + h(t))\,dt + \sigma\,dW_t^T$$

where $h(t)$ is a deterministic function of time (which may depend on $T$) and the process $\{W_t^T\}$ is a Brownian motion under the $T$-terminal measure $\mathbb{Q}_T$. Determine the function $h(t)$ when $T$ is given.

**Solution:** By the change-of-numéraire formula, we have

$$dW_t^T = dW_t^M - \frac{\sigma_T}{\pi_T}\,dt.$$

From the Vasicek bond price formula, the volatility of the price of the bond maturing at time $T$ is found to be

$$\sigma_T = \frac{\partial \pi_T}{\partial r}\sigma = -g(T-t)\pi_T\sigma.$$

Therefore, the Vasicek SDE may be rewritten as

$$\begin{aligned} dr_t &= a(b-r_t)\,dt + \sigma(dW_t^T - g(T-t)\sigma\,dt) = \\ &= (-ar_t + h(t))\,dt + \sigma\,dW_t^T \end{aligned}$$

where $h(t) = ab - g(T-t)\sigma^2$.

(b) Show that, for any given $t \le T_1 \le T_2$, the conditional distribution (given information up to time $t$) of the short rate at time $T_1$ in the Vasicek model, under the $T_2$-terminal measure, is normal. How does this help to compute the value of an interest rate cap?

**Solution:** Notice that the short rate follows an Ornstein-Uhlenbeck process under the $T$-terminal measure. Hence, the conditional distribution (given information up to time $t$) of the short rate at time $T_1$ is normal. By means of the Vasicek bond pricing formula, the condition $P_{T_1}(T_2) < K$ can be rewritten in the form $r_{T_1} > c$ for a certain number $c$. On the basis of the pricing formula found in part b., it is therefore possible to obtain an expression for the value of the bond put at any time $t < T_1$ in terms of the cumulative normal distribution function.

**Problem 6 (Credit Risk)** Suppose Merton's firm value model is used to assess credit risk. The firm value evolves according to

$$\mathrm{d}V_t = V_t\big[\mu \mathrm{d}t + \sigma \mathrm{d}W_t\big],$$

where all parameters are constant, and $W_t$ is a standard Brownian motion under the physical measure $\mathbb{P}$. The risk-free term structure is flat with yield $r$, and the firm has emitted a single zero-coupon bond with notional $F$ and maturity $T$.

(a) Find an explicit expression for the *distance-to-default* that is the distance between the expected value of the asset and the default point.

**Solution:** Default occurs at time $T$ if the firm value falls below $F$. Expected asset price at time $T$ is thus

$$DD^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[V_T] - F = V_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)T} - F,$$
$$DD^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[V_T] - F = V_0 \mathrm{e}^{(r - \frac{1}{2}\sigma^2)T} - F.$$

*Remark:* In practice (especially in the so-caled KMV model), distance-to- default is typically defined in terms of the expected log asset price and log face value and scaled in terms of volatility, i.e.,

$$dd^{\mathbb{Q}} = \frac{\log(V_0 \mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)T}) - \log(F)}{\sigma\sqrt{T}} = d_2.$$

(b) In the lecture we discussed the formula

$$\sigma \frac{\Phi\big(d_1(\sigma)\big)}{E(\sigma)} = \frac{\sigma_E}{V},$$

which relates the firm value volatility to the volatility of equity. Derive this formula.

**Solution:** $E$ is given by the Black-Scholes formula

$$E = V\Phi(d_1) - Fe^{-rT}\Phi(d_2)$$

Notice that $E = E(V, \sigma, F, r, T)$. The dynamics of equity follow from Itô's lemma

$$\begin{aligned}
\mathrm{d}E_t &= \frac{\partial E}{\partial V}\mathrm{d}V_t + \frac{1}{2}\frac{\partial^2 E}{\partial V^2}\mathrm{d}[V]_t \\
&= \frac{\partial E}{\partial V} V_t\big[\mu \mathrm{d}t + \sigma \mathrm{d}W_t\big] + \frac{1}{2}\frac{\partial^2 E}{\partial V^2} V_t^2 \sigma^2
\end{aligned}$$

Since equity stays strictly positive before $T$, it can be written as

$$\mathrm{d}E_t = E_t\big[\mu_E \mathrm{d}t + \sigma_E \mathrm{d}W_t\big]$$

Comparing the volatility parts of these dynamics yields

$$\frac{\partial E}{\partial V} V\sigma = E\sigma_E,$$

where $\frac{\partial E}{\partial V} = \Delta_C = \Phi(d_1)$ is the option delta.