Valuation and Risk Management 2023
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Problem Set 1

Problem 1 (Fundamental Notions and Techniques)

(a) When is a portfolio strategy said to be self-financing? Give a description in words, and also give a mathematical formulation in a continuous-time framework. Justify the mathematical formulation by a limit argument (full rigor not required).

(b) What is an equivalent martingale measure? Explain its importance in quantitative finance. Relate existence and uniqueness of the EMM to economic properties of the market and explain them.

(c) The Feynman-Kac Theorem can be used to relate two important pricing techniques in quantitative finance. State the theorem and explain how it can be applied for option pricing.

(d) State the two fundamental theorems of asset pricing and explain their importance in quantitative finance.

Problem 2 (Stochastic Calculus)

(a) Suppose that the stochastic processes X_t and Y_t satisfy the following system of stochastic differential equations:

$$\begin{aligned} dX_t &= -\tfrac{1}{2} X_t\,dt + Y_t\,dW_t \\ dY_t &= -\tfrac{1}{2} Y_t\,dt - X_t\,dW_t. \end{aligned}$$

Compute $d(X_t^2 + Y_t^2)$.

(b) Suppose that the process X_t satisfies a stochastic differential equation of the form

$$dX_t = \mu_t\,dt + \sigma(t)\,dW_t$$

where μ_t is a continuous semimartingale and $\sigma(t)$ is a deterministic function of time. Prove the following: if $\exp(X_t)$ is a martingale, then

$$\mu_t = -\tfrac{1}{2}\sigma^2(t).$$

(c) Let a process $\{X_t\}$ be defined by

$$X_t = e^{\frac{1}{2}t} \sin W_t.$$

Prove that $\{X_t\}$ is a martingale.

Problem 3 (Black-Scholes Model) Consider the standard Black-Scholes Setting.

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \\ \mathrm{d}M_t &= r M_t \,\mathrm{d}t. \end{aligned}$$

(a) Derive the price of a European put option on stock with maturity T and strike price K from the put-call-parity. Interpret the components of the resulting formula.

(b) Apply the replication recipe to derive the hedging strategy $\phi_t = \phi(t, S_t)$ for the put option.

(c) Derive the pricing kernel and the numeraire portfolio.

Problem 4 (Generic State Space Model) Consider an asset whose price S_t follows a process given by

$$dS_t = \mu_S(t, S_t)\, dt + \sigma_S(t, S_t)\, dW_t.$$

Suppose that there is another traded asset whose price C_t is determined as a continuously differentiable function of t and S_t:

$$C_t = \pi_C(t, S_t).$$

Assume that (i) the price S_t is always positive, (ii) the volatility $\sigma_S(t, S)$ is always positive, and (iii) the relative price C_t/S_t is a strictly increasing function of S_t (in other words, the function $\pi_C(t, S)/S$ is strictly increasing as a function of S for every fixed value of t).

(a) Prove that the market consisting of the two assets S_t and C_t is complete and arbitrage-free. Construct the unique EMM.

Assume now that a third asset is given by the equation

$$dB_t = rB_t\, dt$$

where r is a constant.

(b) State the conditions under which the market is still arbitrage-free.

(c) Assuming that the conditions of the previous part are satisfied, show how the value of the asset B_t can be replicated by a self-financing portfolio consisting of the assets S_t and C_t.

Problem 5 (Option Pricing) The price of an asset S_t follows the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma(t) S_t\, dW_t$$

where μ is a constant and $\sigma(t)$ is a deterministic function of time. The initial value S_0 is given.

(a) Describe the distribution of S_T for a given time $T > 0$. [Hint: compute $d(\log S_t)$.]

(b) Is it possible to derive a closed-form solution for a European call option in this setting along the lines of the Black-Scholes model? What will be different? Explain your answer.

Problem 6 (Option Pricing) Consider the standard Black-Scholes model given by

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t \\ dB_t &= r B_t\, dt. \end{aligned}$$

A geometric Asian digital option with n equispaced sample points is defined by the payoff function

$$C_T = \begin{cases} 1 & \text{if } A_T^n \geq K \\ 0 & \text{if } A_T^n < K \end{cases}$$

where

$$A_T^n = \sqrt[n]{S_{t_1} S_{t_2} \cdots S_{t_n}}\,, \qquad t_i = i\,\frac{T}{n}, \quad i = 1, \ldots, n.$$

(a) Show that, under the risk-neutral measure, the random variable $L_T^n = \log A_T^n$ follows a normal distribution, and give an expression for its mean and its variance.

(b) Give an explicit formula, in terms of the cumulative normal distribution function, for the option value in case $n = 2$ and in case $n = 3$.

Problem 7 (Generic State Space Model) Assume the following option pricing model with stochastic interest rates under the real-world measure.

$$
\begin{aligned}
\mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma_s S_t \,\mathrm{d}W_{1,t} \\
\mathrm{d}M_t &= r_t M_t \,\mathrm{d}t \\
\mathrm{d}r_t &= a(b - r_t)\,\mathrm{d}t + \sigma_r \sqrt{r_t}(\rho \,\mathrm{d}W_{1,t} + \sqrt{1-\rho^2}\,\mathrm{d}W_{2,t})
\end{aligned}
$$

where $\mu, a, b \in \mathbb{R}$ are real constants, $\sigma_r, \sigma_s > 0$, $\rho \in (-1, 1)$, and W_1 and W_2 are two independent standard Brownian motions.

(a) Show that the model is arbitrage free yet incomplete.

Suppose that another asset with price V_t is added to this model, and that W_t satisfies the stochastic differential equation

$$
\mathrm{d}V_t = V_t[r_t + \pi(\mu - r_t)]\,\mathrm{d}t + V_t \pi \sigma_s \,\mathrm{d}W_{1,t}
$$

where π is a constant.

(b) Show that the asset V_t can be replicated by following a self-financing trading strategy using the stock S_t and the money market account M_t. Determine the replicating strategy explicitly in terms of the state variables M_t, S_t, and V_t.

(d) Why is it possible to replicate V_t despite the fact that the market is incomplete?

(e) State an interpretation of the asset V_t and the parameter π.

(f) Explain why the asset V_t can be taken as a numéraire and relate the parameter π to three different equivalent martingale measures in the lecture and the numéraires they are associated with.

(g) Suppose now you want to price a derivative with payoff $C_T = F(S_T, r_T)$. Write down the PDE the derivative has to satisfy. What problem does occur in this setting?