Valuation and Risk Management 2023
Tilburg School of Economics and Management
Christoph Hambel
Problem Set 1

Problem 1 (Fundamental Notions and Techniques)

(a) When is a portfolio strategy said to be self-financing? Give a description in words, and also give a mathematical formulation in a continuous-time framework. Justify the mathematical formulation by a limit argument (full rigor not required). **Solution:** A trading strategy is said to be self-financing if trading does neither generate nor destroy money. Changes in the portfolio wealth are solely driven by changes in asset pricing.

Starting in discrete time, one can define a self-financing strategy as follows

$$\phi_t' Y_t = \phi_{t-1}' Y_t$$

where Y denotes the vector of asset prices. One can show that this is equivalent to

$$V_T = V_0 + \sum_{j=0}^{n-1} \phi_{t_j}' \, \Delta Y_{t_{j+1}}$$

Taking the limit to continuous time yields

$$V_T = V_0 + \int_0^T \phi_t' \, \mathrm{d}Y_t$$

or $\mathrm{d}V_t = \phi_t' \, \mathrm{d}Y_t$.

(b) What is an equivalent martingale measure? Explain its importance in quantitative finance. Relate existence and uniqueness of the EMM to economic properties of the market and explain them.

Solution: An equivalent martingale measure is a probability measure equivalent to $\mathbb{P}$ under which deflated asset prices (with an appropriate numéraire N are martingales) that is $(Y_t/N_t)_{t\geq 0}$ is a martingale under $\mathbb{Q}_N$. This property reduces the pricing problem of assets to the calculation of (conditional) expectations if such an EMM exists.

An EMM (for a given numéraire) exists if and only if the market is free of arbitrage, i.e., if you cannot make money out of nothing without incurring risks.

In an arbitrage-free market for every given numéraire the EMM is uniquely determined if and only if the market is complete, i.e., if one can replicate every riky cash-flow by traded assets.

(c) The Feynman-Kac Theorem can be used to relate two important pricing techniques in quantitative finance. State the theorem and explain how it can be applied for option pricing.

Solution: Consider the following parabolic partial differential equation

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X^{\mathbb{Q}}(t,x) + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X(t,x)\sigma_X(t,x)'\Big) + f(t,x) = r(t,x)\pi_C$$

subject to the terminal condition $\pi_C(T,x) = F(x)$. Then, the solution can be written as a conditional expectation

$$\pi_C(t,x) = \mathbb{E}_{t,x}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(\tau,X_\tau)\,d\tau} f(s,X_s)\mathrm{d}s + e^{-\int_t^T r(\tau,X_\tau)\,d\tau} F(X_T)\right]$$

under $\mathbb{Q}$ such that X is an Itô process driven by the equation

$$\mathrm{d}X = \mu_X^{\mathbb{Q}}(t,X)\,\mathrm{d}t + \sigma_X(t,X)\,\mathrm{d}W^{\mathbb{Q}},$$

with $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$. It connects the pricing problem to the solution of the PDE.

(d) State the two fundamental theorems of asset pricing and explain their importance in quantitative finance.

Problem 2 (Stochastic Calculus)

(a) Suppose that the stochastic processes X_t and Y_t satisfy the following system of stochastic differential equations:

$$dX_t = -\tfrac{1}{2}X_t\,dt + Y_t\,dW_t$$
$$dY_t = -\tfrac{1}{2}Y_t\,dt - X_t\,dW_t.$$

Compute $d(X_t^2 + Y_t^2)$.

Solution: Applying Ito's lemma yields

$$dX_t^2 = 2X_t[-\tfrac{1}{2}X_t\,dt + Y_t\,dW_t] + Y_t^2 dt$$
$$dY_t^2 = 2Y_t[-\tfrac{1}{2}Y_t\,dt - X_t\,dW_t] + X_t^2 dt$$

Therefore,

$$d(X_t^2 + Y_t^2) = 0$$

Consequently, the process $X_t^2 + Y_t^2$ is constant.

(b) Suppose that the process X_t satisfies a stochastic differential equation of the form

$$dX_t = \mu_t \, dt + \sigma(t) \, dW_t$$

where μ_t is a continuous semimartingale and $\sigma(t)$ is a deterministic function of time. Prove the following: if $\exp(X_t)$ is a martingale, then

$$\mu_t = -\tfrac{1}{2}\sigma^2(t).$$

Solution: Applying Ito's lemma yields

$$\begin{aligned} d\exp(X_t) &= \exp(X_t)[\mu_t \, dt + \sigma(t) \, dW_t] + \frac{1}{2}\exp(X_t)\sigma(t)\mathrm{d}t \\ &= \exp(X_t)\big[\mu_t \, dt + \frac{1}{2}\sigma(t)\sigma(t) \, dW_t\big]. \end{aligned}$$

An Ito process is a martingale if and only if its drift term is zero, i.e.,

$$\mu_t + \frac{1}{2}\sigma(t) = 0$$

which proves the claim.

(c) Let a process $\{X_t\}$ be defined by

$$X_t = e^{\frac{1}{2}t} \sin W_t.$$

Prove that $\{X_t\}$ is a martingale.

Solution: Applying Ito's product rule yields

$$dX_t = \sin W_t e^{\frac{1}{2}t} \tfrac{1}{2} dt + e^{\frac{1}{2}t} d(\sin W_t)$$

Now, we need to calculate $d(\sin W_t)$. Consequently, we apply Ito's lemma on the function $f(x) = \sin(x)$. Since $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$ we obtain

$$\begin{aligned} dX_t &= \sin W_t e^{\frac{1}{2}t} \tfrac{1}{2} dt + e^{\frac{1}{2}t}\big[\cos W_t \mathrm{d}W_t - \tfrac{1}{2}\sin W_t dt\big] \\ &= e^{\frac{1}{2}t} \cos W_t \mathrm{d}W_t \end{aligned}$$

Since, the drift rate is zero, this process is indeed a martingale.

Problem 3 (Black-Scholes Model) Consider the standard Black-Scholes Setting.

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \\ \mathrm{d}M_t &= r M_t \,\mathrm{d}t. \end{aligned}$$

(a) Derive the price of a European put option on stock with maturity T and strike price K from the put-call-parity. Interpret the components of the resulting formula.

Solution: The put-call-parity reads

$$P_t = C_t - S_t + K\mathrm{e}^{-rt}$$

Substitute the Black-Scholes formula for a European call option into the PC parity:

$$\begin{aligned} P_t &= S_t\Phi(d_1) - K\mathrm{e}^{-rt}\Phi(d_2) - S_t + K\mathrm{e}^{-rt} \\ &= K\mathrm{e}^{-rt}[1 - \Phi(d_2)] - S_t[1 - \Phi(d_1)] \\ &= K\mathrm{e}^{-rt}\Phi(-d_2) - S_t\Phi(-d_1) \end{aligned}$$

where d_1 and d_2 is the same as for the call option.

(b) Apply the replication recipe to derive the hedging strategy $\phi_t = \phi(t, S_t)$ for the put option.

Solution: 1st step is part (a). Notice that $X = Y$.

2nd step: put price volatility:

$$\sigma_P(t,x) = \nabla P(t,x)\,\sigma_X(t,x) = -\sigma S_t \Phi(-d_1).$$

3rd step: Dynamic trading strategy

$$[\sigma_Y \quad \pi_Y] = \begin{bmatrix} S_t\sigma & S \\ 0 & M_t \end{bmatrix}.$$

Therefore, the trading strategy ϕ satisfies the following linear system:

$$\begin{bmatrix} -\sigma S_t\Phi(-d_1) & P_t \end{bmatrix} = \phi' \begin{bmatrix} S_t\sigma & S \\ 0 & M_t \end{bmatrix}$$

Therefore,

$$\phi' = \begin{bmatrix} S_t\sigma & S \\ 0 & M_t \end{bmatrix}^{-1} \begin{bmatrix} -\sigma S_t\Phi(-d_1) & P_t \end{bmatrix}$$

Straightforward calculations deliver

$$\phi' = \begin{bmatrix} -\Phi(-d_1) \\ K\Phi(-d_2) \end{bmatrix}.$$

(c) Derive the pricing kernel and the numeraire portfolio.

Solution: There are various solution approaches. The RN process satisfies

$$\mathrm{d}\theta_t = \theta_t \lambda \mathrm{d}W$$

where $\lambda = \frac{\mu - r}{\sigma}$. In particular, the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is completely described by

$$\theta_t = \exp\Big(-\tfrac{1}{2}\lambda^2 t - \lambda W_t \Big)$$

Consequently, the pricing kernel must be $K_t = \theta_t \mathrm{e}^{-rt}$ (combine change of measure and discounting at r). The (wealth generated by the) numéraire portfolio is the inverse of the pricing kernel, i.e.,

$$V_t = \frac{1}{K_t} = \exp\Big(rt + \tfrac{1}{2}\lambda^2 t + \lambda W_t \Big) \tag{1}$$

Notice that this wealth is generated by a portfolio which invests the fraction $\pi = \frac{\lambda}{\sigma}$ of wealth in the stock and the remainder in the risk-free asset. Wealth dynamics of this strategy are

$$\mathrm{d}V_t^\pi = V_t^\pi [r + \pi(\mu - r)\mathrm{d}t + \sigma\pi \mathrm{d}W_t]$$

and the wealth is

$$V_t^\pi = \exp\Big([r + \pi(\mu - r)]t - \tfrac{1}{2}\sigma^2\pi^2 t + \sigma\pi W_t \Big) \tag{2}$$

Comparing (2) and (1) shows $\pi = \frac{\lambda}{\sigma}$. One can alternatively solve the optimization problem for the log-investor to determine the portfolio strategy π.

Problem 4 (Generic State Space Model) Consider an asset whose price S_t follows a process given by

$$dS_t = \mu_S(t, S_t)\, dt + \sigma_S(t, S_t)\, dW_t.$$

Suppose that there is another traded asset whose price C_t is determined as a continuously differentiable function of t and S_t:

$$C_t = \pi_C(t, S_t).$$

Assume that (i) the price S_t is always positive, (ii) the volatility $\sigma_S(t,S)$ is always positive, and (iii) the relative price C_t/S_t is a strictly increasing function of S_t (in other words, the function $\pi_C(t,S)/S$ is strictly increasing as a function of S for every fixed value of t).

(a) Prove that the market consisting of the two assets S_t and C_t is complete and arbitrage-free. Construct the unique EMM.

Solution: The market is described in terms of one state variable (S_t), one driving Brownian motion, and two traded assets (S_t and C_t). We have

$$\begin{bmatrix} \sigma_Y & \pi_Y \end{bmatrix} = \begin{bmatrix} \sigma_S(t,S) & S \\ \sigma_C(t,S) & \pi_C(t,S) \end{bmatrix}$$

where Y_t is the vector of asset prices, and $\sigma_C = (\partial \pi_C/\partial S)\sigma_S$ by Itô's rule. The market is complete and arbitrage-free if and only if the above matrix is invertible for all t and S, or in other words, if and only if its determinant is zero.

$$\begin{aligned} \sigma_S(t,S)\pi_C(t,S) - S\sigma_C(t,S) &\neq 0 \quad \text{for all } t \text{ and } S \\ \pi_C(t,S) - S\frac{\partial \pi_C}{\partial S}(t,S) &\neq 0 \quad \text{for all } t \text{ and } S \end{aligned}$$

because the common factor $\sigma_S(t,S)$ is always positive. Since the function $\pi_C(t,S)/S$ is strictly increasing as a function of S, its partial derivative with respect to S is positive:

$$0 < \frac{\partial(\pi_C(t,S)/S)}{\partial S} = \frac{1}{S}\frac{\partial \pi_C(t,S)}{\partial S} - \frac{\pi_C(t,S)}{S^2}.$$

This implies the condition above (multiply by S^2).

Assume now that a third asset is given by the equation

$$dB_t = rB_t\,dt$$

where r is a constant.

(b) State the conditions under which the market is still arbitrage-free.

Solution: The condition for the extended market to be arbitrage-free is that the equation

$$\begin{bmatrix} \mu_S \\ \mu_C \\ rB \end{bmatrix} = \begin{bmatrix} \sigma_S & S \\ \sigma_C & \pi_C \\ 0 & B \end{bmatrix} \begin{bmatrix} \lambda \\ \tilde{r} \end{bmatrix}$$

(with $B = B_0 e^{rt}$) admits a solution $(\lambda, \tilde{r})$. From the first and the third equation we get $\tilde{r} = r$ and $\lambda = (\mu_S - rS)/\sigma_S$. The condition to be fulfilled is therefore

$$\mu_C - r\pi_C = \sigma_C \frac{\mu_S - rS}{\sigma_S} = (\mu_S - rS)\frac{\partial \pi_C}{\partial S}\,.$$

We now apply Ito's lemma to determine the drift rate of $\pi_C(t, S)$:

$$d\pi_C = \frac{\partial \pi_C}{\partial t}dt + \frac{\partial \pi_C}{\partial S}dS + \tfrac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2}\sigma_S^2 dt = 0$$

Comparing these dynamics with

$$d\pi_C = \mu_C dt + \sigma_C dW$$

yields $\mu_C = \frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S}\mu_S + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2}\sigma_S^2$. Substituting μ_C into the condition above yields

$$\frac{\partial \pi_C}{\partial t} + rS\,\frac{\partial \pi_C}{\partial S} + \tfrac{1}{2}\sigma_S^2\,\frac{\partial^2 \pi_C}{\partial S^2} = r\pi_C.$$

(c) Assuming that the conditions of the previous part are satisfied, show how the value of the asset B_t can be replicated by a self-financing portfolio consisting of the assets S_t and C_t.

Solution: Apply the replication recipe:

$$\begin{bmatrix} \sigma_B & \pi_B \end{bmatrix} = \begin{bmatrix} \phi_S & \phi_C \end{bmatrix} \begin{bmatrix} \sigma_S & S \\ \sigma_C & \pi_C \end{bmatrix}.$$

Since $\sigma_B = 0$ and $\pi_B = B_0 e^{rt}$, we get

$$0 = \phi_S \sigma_S + \phi_C \sigma_C = \phi_S \sigma_S + \phi_C \frac{\partial \pi_C}{\partial S}\sigma_S$$

so that $\phi_S = -(\partial \pi_C/\partial S)\phi_C$, and

$$e^{rt}B_0 = \phi_S S + \phi_C \pi_C = \phi_C\big(-(\partial \pi_C/\partial S)S + \pi_C\big).$$

We find $\phi_C = e^{rt}B_0/\big(\pi_C - (\partial \pi_C/\partial S)S\big)$ and $\phi_S = -(\partial \pi_C/\partial S)\phi_C$.

Problem 5 (Option Pricing) The price of an asset S_t follows the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma(t) S_t\, dW_t$$

where μ is a constant and $\sigma(t)$ is a deterministic function of time. The initial value S_0 is given.

(a) Describe the distribution of S_T for a given time $T > 0$. [Hint: compute $d(\log S_t)$.]

Solution: Applying Ito's rule yields

$$d \log S = \mu dt - \frac{1}{2}\sigma(t)^2 dt + \sigma(t) dW_t$$

Integrating these dynamics yields

$$\log S_t = \log S_0 + \int_0^t \mu ds - \int_0^t \frac{1}{2}\sigma(s)^2 ds + \int_0^t \sigma(s) dW_s$$

Consequently, $\log S_T$ is normally distributed with mean $M(0,T) = \log S_0 + \mu T + \frac{1}{2}\int_0^T \sigma(s)^2 ds$ and variance $\Sigma^2(0,T) = \int_0^T \sigma^2(s)ds$. Hence S_T is lognormally distributed w.r.t. the parameters $M(0,T)$ and $\Sigma^2(0,T)$.

(b) Is it possible to derive a closed-form solution for a European call option in this setting along the lines of the Black-Scholes model? What will be different? Explain your answer.

Solution: Yes. We know the distribution of S_T under $\mathbb{P}$ and can derive it under $\mathbb{Q}$ and $\mathbb{Q}_S$. The distribution will not change, but the parameters will change. Nevertheless, it will be possible to evaluate the probabilities

$$\mathbb{Q}^S(S_T > K), \qquad \mathbb{Q}(S_T > K)$$

along the lines of the BS-model since S_T is lognormal.

Problem 6 (Option Pricing) Consider the standard Black-Scholes model given by

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$
$$dB_t = r B_t\, dt.$$

A geometric Asian digital option with n equispaced sample points is defined by the payoff function

$$C_T = \begin{cases} 1 & \text{if } A_T^n \geq K \\ 0 & \text{if } A_T^n < K \end{cases}$$

where

$$A_T^n = \sqrt[n]{S_{t_1} S_{t_2} \cdots S_{t_n}}\,, \qquad t_i = i\,\frac{T}{n}, \quad i = 1, \ldots, n.$$

(a) Show that, under the risk-neutral measure, the random variable $L_T^n = \log A_T^n$ follows a normal distribution, and give an expression for its mean and its variance.

Solution: Under $\mathbb{Q}$, the stock dynamics are obviously

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}$$

Therefore,

$$\begin{aligned} d\log S_t &= (r - \tfrac{1}{2}\sigma^2)\,dt + \sigma\,dW_t^{\mathbb{Q}} \\ \log S_{t_i} &= \log S_0 + (r - \tfrac{1}{2}\sigma^2)t_i + \sigma W_{t_i}^{\mathbb{Q}} \end{aligned}$$

This shows that $\log S_{t_i}$ is affine in $W^{\mathbb{Q}}$. Since the vector $(W_{t_1}^{\mathbb{Q}}, \ldots, W_{t_n}^{\mathbb{Q}})$ follows a multivariate normal distribution under $\mathbb{Q}$, the vector $(\log S_{t_1}, \ldots, \log S_{t_n})$ is multivariate normally distributed as well.

Since $\log A_T^n = \frac{1}{n}\sum_{i=1}^n \log S_{t_i}$, we obtain

$$\log A_T^n = \frac{1}{n}\sum_{i=1}^n \left[\log S_0 + (r - \tfrac{1}{2}\sigma^2)i\frac{T}{n} + \sigma W_{i\frac{T}{n}}^{\mathbb{Q}}\right]$$

Obviously, the mean of $\log A_T^n$ is

$$\begin{aligned} \mathbb{E}[\log A_T^n] &= \log S_0 + (r - \tfrac{1}{2}\sigma^2)\frac{T}{n}\frac{1}{n}\sum_{i=1}^n i \\ &= \log S_0 + (r - \tfrac{1}{2}\sigma^2)T\frac{n+1}{2n} \end{aligned}$$

Calculating the variance

$$var(\log A_T^n) = var\Big(\frac{1}{n}\sum_{i=1}^n \sigma W_{t_i}^{\mathbb{Q}}\Big) = \frac{\sigma^2}{n^2} var\Big(\sum_{i=1}^n W_{t_i}^{\mathbb{Q}}\Big)$$

Notice that $cov(W_{t_i}^{\mathbb{Q}}, W_{t_j}^{\mathbb{Q}}) = \min(t_i, t_j)$. Consequently,

$$var(\log A_T^n) = \frac{\sigma^2}{n^2}\sum_{i=1}^n\sum_{j=1}^n \min(t_i, t_j) = \frac{\sigma^2}{n^2}\frac{T}{n}\sum_{i=1}^n\sum_{j=1}^n \min(i, j)$$

(b) Give an explicit formula, in terms of the cumulative normal distribution function, for the option value in case $n = 2$ and in case $n = 3$.

Solution: The price of the digital option

$$C_t = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}[1_{\{\log A_T^n \geq \log K\}}] = \mathrm{e}^{-rT}\mathbb{Q}(\log A_T^n \geq \log K)$$

Normalizing and standardizing yields

$$C_t = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}[1_{\{\log A_T^n \geq \log K\}}] = \mathrm{e}^{-rT}\mathbb{Q}\Big(Z_T \geq \frac{\log K - \mathbb{E}[\log A_T^n]}{\sqrt{var(\log A_T^n)}}\Big),$$

where $Z_T \sim (0, 1)$. Consequently, $C_t = \mathrm{e}^{-rT}\Phi(d)$ with

$$d = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T\frac{n+1}{2n}}{\frac{\sigma}{n}\sqrt{\frac{T}{n}\sum_{i=1}^n\sum_{j=1}^n \min(i,j)}}$$

For $n = 2$:

$$d = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)\frac{3T}{4}}{\sigma\sqrt{\frac{5T}{8}}}$$

For $n = 3$:

$$d = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)\frac{2T}{3}}{\sigma\sqrt{\frac{14T}{27}}}$$

Problem 7 (Generic State Space Model) Assume the following option pricing model with stochastic interest rates under the real-world measure.

$$\begin{aligned}
\mathrm{d}S_t &= \mu S_t\,\mathrm{d}t + \sigma_s S_t\,\mathrm{d}W_{1,t} \\
\mathrm{d}M_t &= r_t M_t\,\mathrm{d}t \\
\mathrm{d}r_t &= a(b - r_t)\,\mathrm{d}t + \sigma_r\sqrt{r_t}(\rho\,\mathrm{d}W_{1,t} + \sqrt{1-\rho^2}\,\mathrm{d}W_{2,t})
\end{aligned}$$

where $\mu, a, b \in \mathbb{R}$ are real constants, $\sigma_r, \sigma_s > 0$, $\rho \in (-1, 1)$, and W_1 and W_2 are two independent standard Brownian motions.

(a) Show that the model is arbitrage free yet incomplete.

Solution: NA criterion: $\pi_y = [S \;\; M]'$, $\mu_y = [S\mu \;\; Mr]'$, $\sigma_y =$

$$\sigma_y\lambda + \pi_y\widehat{r} = \mu_y$$

leads to the system

$$\sigma_s S\lambda_1 + Sr = \mu S$$
$$M\widehat{r} = Mr,$$

which has a solution $[\lambda_1 \quad \lambda_2 \quad \widehat{r}]'$ with $\lambda_1 = \frac{\mu - r}{\sigma_s}$, $\widehat{r} = r$, and λ_2 can be chosen arbitrarily. Hence, there is a risk-neutral measure, but the change of measure is not uniquely determined. According to the FTAPs, the market is free of arbitrage yet incomplete.

Suppose that another asset with price V_t is added to this model, and that W_t satisfies the stochastic differential equation

$$\mathrm{d}V_t = V_t[r_t + \pi(\mu - r_t)]\,\mathrm{d}t + V_t\pi\sigma_s\,\mathrm{d}W_{1,t}$$

where π is a constant.

(b) Show that the asset V_t can be replicated by following a self-financing trading strategy using the stock S_t and the money market account M_t. Determine the replicating strategy explicitly in terms of the state variables M_t, S_t, and V_t.

Solution: Replication recipe

$$V_t = \phi_1 S + \phi_2 M$$
$$V_t\pi\sigma_s = \phi_1 S\sigma_s$$

Consequently, $\phi_1 = \frac{V\pi}{S}$, $\phi_2 = \frac{V - \phi_1 S}{M}$

(d) Why is it possible to replicate V_t despite the fact that the market is incomplete?
Solution: Because V is a linear combination of the MMA and the stock and not driven by the second SBM.

(e) State an interpretation of the asset V_t and the parameter π.

Solution: V is a portfolio where the investor invests a a fraction π of her wealth in the stock and the residual in the MMA.

(f) Explain why the asset V_t can be taken as a numéraire and relate the parameter π to three different equivalent martingale measures in the lecture and the numéraires they are associated with.

Solution: V is a tradeable asset that stays strictly positive.

(a) $\pi = 0$: $N_t = M_t$, $\mathbb{Q} \sim \mathbb{P}$

(b) $\pi = 1$: $N_t = S_t$, $\mathbb{Q}_S \sim \mathbb{P}$

(c) $\pi = \frac{\mu - r}{\sigma_s}$, N is the numeraire portfolio, and $\mathbb{Q}_N = \mathbb{P}$

(g) Suppose now you want to price a derivative with payoff $C_T = F(S_T, r_T)$. Write down the PDE the derivative has to satisfy. What problem does occur in this setting?

Solution:

$$C_t + C_S r S + \frac{1}{2} C_{SS} S^2 \sigma_s^2 + C_r \widehat{a}(\widehat{b} - r) + \frac{1}{2} C_{rr} r \sigma_r^2 + C_{rS} S \sqrt{r} \sigma_r \sigma_s \rho = rC$$

Problem: the drift rate of r is not uniquely determined under $\mathbb{Q}$ due to market incompleteness.