# Part I

# Introduction to Financial Modeling

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# Time

- Discrete time with time horizon $T$:

$$t \in \{0, \Delta t, 2\Delta t, \ldots, (n-1)\Delta t, \underbrace{n\Delta t}_{=T}\} = \{i\Delta t \mid i = 0, \ldots, n\}$$

- Continuous time as a limit of discrete time ($\Delta t \to 0$ as $n \to \infty$):

$$t \in [0, T]$$

# Modeling in Discrete Time: First Idea

- Risk-free asset (bond) paying a constant interest rate:

$$B_{t+\Delta t} = B_t(1 + r \cdot \Delta t) \qquad \Longleftrightarrow \qquad \frac{\Delta B_{t+\Delta t}}{B_t} = r \cdot \Delta t$$

- Risky asset (stock):

$$S_{t+\Delta t} = S_t(1 + \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}), \qquad \nu_{t+\Delta t} \sim_{i.i.d.} (0, 1)$$

- Return:

$$\frac{\Delta S_{t+\Delta t}}{S_t} = \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

**Problem:** Returns are not necessarily bounded from below by -1 and thus asset prices can be negative.

# Log Returns



- Way out? $\rightarrow$ Model log returns, $L_t$, and take the exponential:

$$S_{t+\Delta t} = S_t \mathrm{e}^{\Delta L_{t+\Delta t}}$$

- Risk-free asset (bond):

$$B_{t+\Delta t} = B_t \mathrm{e}^{r \cdot \Delta t} \quad \Longleftrightarrow \quad r\Delta t = \ln\left(\frac{B_{t+\Delta t}}{B_t}\right) = \Delta \ln B_{t+\Delta t}$$

- Risky asset (stock):

$$\Delta L_{t+\Delta t} = \ln(S_{t+\Delta t}) - \ln(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

- Now, we take the limit to continuous time, i.e., we increase the number of periods ($n \rightarrow \infty$) while keeping the time horizon constant, i.e., $\Delta t = \frac{T}{n} \rightarrow 0$.

# Log Returns

$$S_T = S_0 \prod_{i=0}^{n-1} \mathrm{e}^{\Delta L_{(i+1)\Delta t}}$$

$$= S_0 \exp\left\{ \sum_{i=0}^{n-1} \left[ \left( \mu - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma \cdot \nu_{(i+1)\Delta t} \cdot \sqrt{\Delta t} \right] \right\}$$

$$= S_0 \exp\left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{\Delta t} \cdot \sum_{i=1}^{n} \nu_{i\Delta t} \right\}$$

$$= S_0 \exp\left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{T} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i\Delta t} \right\}$$

According to the CLT: $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i\Delta t} \to_d Z_T \sim \mathcal{N}(0,1)$ as $n \to \infty$, i.e.,

$$S_T \to_d S_0 \exp\left\{ \left( \mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{T} \cdot Z_T \right\}$$

- In the limit, the log return is normally distributed:

$$L_T = L_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma \cdot \sqrt{T} \cdot Z_T$$

- Consequently, in the limit $S_T$ is log-normally distributed with

$$\text{mean:} \qquad \mathbb{E}[S_T] = S_0 e^{\mu \cdot T}$$
$$\text{variance:} \quad \text{var}(S_T) = S_0^2 e^{2\mu \cdot T}[e^{\sigma^2 T} - 1]$$

- Does this mean that any discrete-time model converges to a log-normal distribution?

- How can we model asset prices in continuous time?

# Trading in Discrete Time



- Assume that there is a frictionless financial market (i.e., no taxes, no transaction costs, no short-selling constraints, ...)
- Throughout the lecture we will be using vector notation:

$$m : \text{number of basic assets}$$

$$Y_t : m\text{-dimensional vector of asset prices at time } t$$

$$\phi_t : \text{vector of number of units of assets held at time } t$$

- Portfolio value generated by the *portfolio strategy* (or *trading strategy*) $\phi$:

$$V_t = \phi_t' Y_t.$$

- A portfolio strategy $\phi$ is *self-financing* if trading neither generates nor destroys money, i.e.,

$$\phi_{t-\Delta t}' Y_t = \phi_t' Y_t.$$

# Trading in Discrete Time



- Suppose that rebalancing takes place at times $0 < t_1 < \cdots < t_n = T$, i.e., $t_j = j\Delta t$.

$$
\begin{aligned}
V_T &= V_0 + \sum_{j=0}^{n-1} (V_{t_{j+1}} - V_{t_j}) && \text{(telescope rule)} \\
&= V_0 + \sum_{j=0}^{n-1} \phi'_{t_j} (Y_{t_{j+1}} - Y_{t_j}) && \text{(self-financing portfolio)} \\
&= V_0 + \sum_{j=0}^{n-1} \phi'_{t_j} \, \Delta Y_{t_{j+1}}.
\end{aligned}
$$

- The sum $\sum_{j=0}^{n-1} \phi'_{t_j} \, \Delta Y_{t_{j+1}}$ converges in some sense to the stochastic integral $\int_0^T \phi'_t \, \mathrm{d} Y_t$ even if the integrator is of infinite variation.
- The continuous-time version of self-financing is $V_T = V_0 + \int_0^T \phi'_t \, \mathrm{d} Y_t$.

# From Discrete Time to Continuous Time



- We need adequate tools for modeling asset prices in continuous time that can be interpreted along the lines of

$$(1) \quad \frac{\Delta B_{t+\Delta t}}{B_t} = r \cdot \Delta t$$

$$(2) \quad \frac{\Delta S_{t+\Delta t}}{S_t} = \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

and that preserve the limit distribution of the stock return.

- Replace (1) by an ODE and (2) by an SDE:

$$(1') \quad \frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t$$

$$(2') \quad \frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

- Replace the self-financing condition $\phi'_{t-\Delta t} Y_t = \phi'_t Y_t$ by $V_T = V_0 + \int_0^T \phi'_t \, \mathrm{d}Y_t$ for an adequately defined stochastic integral.

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# Stochastic Processes

- Consider a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$
  - $\Omega$ denotes the state space.
  - $\mathcal{A} \subset 2^{\Omega}$ denotes a sigma algebra that contains all events for which probabilities can be assigned.
  - $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration, which models the set of information available at time $t$.
  - $\mathbb{P} : \mathcal{A} \to [0,1]$ is a probability measure, which we refer to as *real-world* probability measure.
- A stochastic process $X$ is a collection of random variables $(X_t)_{t\geq 0}$ indexed by time.
- **Remarks:**
  - Throughout the course, we assume that all processes are continuous (i.e., "no jumps" a.s.) and adapted (i.e., "realization $X_t$ is known at time $t$"). Formulas become more involved if we relax this assumption.
  - I will avoid technical terms (e.g., measurability, integrability), but focus on economic interpretations. I will rather assume that all processes satisfy all relevant conditions.

**Definition (Brownian Motion)**

A one-dimensional (standard) *Brownian motion* (aka *Wiener Process*) is a stochastic process $W = (W_t)_{t \geq 0}$ such that $W_0 = 0$ a.s. and

- $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$ (*stationary increments*).
- $W_t - W_s$ is independent of $W_u - W_v$ for $0 \leq v < u \leq s < t$ (*independent increments*).

- A $k$-dimensional standard Brownian motion $W = (W_1, \ldots, W_k)$ is a $k$-dimensional vector of independent Brownian motions.
- Notice that the paths of a Brownian motion are continuous (a.s.) but nowhere differentiable. In particular, the paths of Brownian motion have infinite length on any interval ("infinite variation").

# Martingales

## Definition (Martingale)

A stochastic process $Z = (Z_t)_{t \geq 0}$ is said to be a *martingale* if "the best estimate of the future is the present", i.e.,

$$E_s[Z_t] = Z_s \qquad t \geq s$$

- Martingales relate to "fair games" and are often thought of as "purely stochastic" processes, that is, containing no trend or being constant in expectation..
- Example: Brownian motion is a martingale.
- There are many generalizations of martingales, e.g.,
  - Submartingales ("non-decreasing in expectation")
  - Supermartingales ("non-increasing in expectation")
  - Local martingales ("if stopped process is a martingale")
  - Semimartingales ("local martingale + process of finite variation")

# Itô Integral

- The stochastic integral (a.k.a. Itô integral) is defined by

$$\int_0^T X_t \, \mathrm{d}Z_t = \lim_{n\to\infty} \sum_{j=0}^{n} X_{t_j}(Z_{t_{j+1}} - Z_{t_j})$$

  where $Z$ is a semimartingale, $X$ is an adapted process, and the stochastic limit is taken in the sense of refining partitions (i.e., intermediate points $t_0, t_1, \ldots, t_n$ become more and more dense on the interval $[0, T]$ as $n$ tends to infinity).
- The construction of the limit and prove of convergence is not trivial, since in general the integrator is of infinite variation.
- Such a limit does not necessarily exist pathwise.
- Note: by contrast to the Riemann-Stieltjes integral, the integrand is evaluated at the left end $t_j$.
- The stochastic integral is itself a random variable.

## Definition (Stochastic Differential Equation)

Let $W$ be a standard Brownian motion. An expression of the form

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t$$

for given functions $\mu(t, X_t)$ (*drift*) and $\sigma(t, X_t)$ (*volatility*) is called a *stochastic differential equation* (SDE) driven by Brownian motion and should be understood as a short-hand notation for the integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s.$$

- If the drift $\mu(t, X_t)$ is zero, then the solution is a martingale.
- This definition can be generalized to SDEs driven by jump processes (e.g., Poisson processes).

# Quadratic (Co-)Variation

- Let $X$, $Y$ be two real-valued stochastic processes, then their *quadratic covariation process* is defined as

$$[X, Y]_t = \lim_{\Delta t \to 0} \sum_{j=0}^{t} (X_{t_{j+1}} - X_j)(Y_{t_{j+1}} - Y_j)$$

- The *quadratic variation process* of $X$ is defined by

$$[X]_t = [X, X]_t$$

- Rules for quadratic (co)-variation:
  - linearity in both arguments
  - $[X, g] = 0$ if $g$ is a continuous function of bounded variation
  - $\mathrm{d}[W_1, W_2] = \rho \, \mathrm{d}t$ for BMs with correlation coefficient $\rho$; $\mathrm{d}[W] = \mathrm{d}t$
  - if $\mathrm{d}X = \mu_X \, \mathrm{d}t + \sigma_X \, \mathrm{d}W_1$ and $\mathrm{d}Y = \mu_Y \, \mathrm{d}t + \sigma_Y \, \mathrm{d}W_2$, then

$$\mathrm{d}[X, Y] = \sigma_X \sigma_Y \rho \, \mathrm{d}t, \qquad \mathrm{d}[X] = \sigma_X^2 \, \mathrm{d}t$$

# Itô's Lemma: Univariate Versions



**Theorem (Itô's Lemma for continuous semimartingales)**

Let $X$ be a continuous real-valued semimartingale, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)\,\mathrm{d}t + \frac{\partial}{\partial x} f(t, X_t)\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t)\,\mathrm{d}[X, X]_t.$$

**Theorem (Itô's Lemma for Itô processes)**

Let $X$ be an Itô process $\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}W_t$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \left[\frac{\partial}{\partial t} f(t, X_t) + \frac{\partial}{\partial x} f(t, X_t)\mu_X + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t)\,\sigma_X^2\right]\mathrm{d}t + \frac{\partial}{\partial x} f(t, X_t)\sigma\,\mathrm{d}W_t.$$

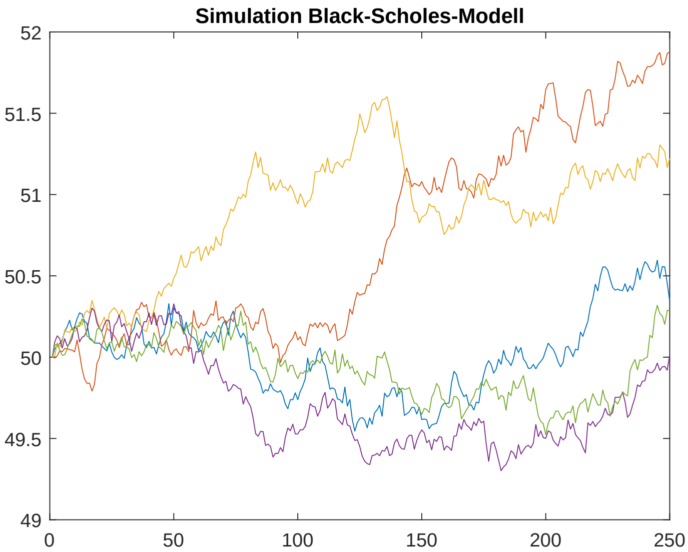
# Problem: Black Scholes Model

**Problem:** Derive the stock price in the Black-Scholes model and show that it is strictly positive almost surely.

**Solution:**

# Problem: Black Scholes Model

# Geometric Brownian Motion



# Itô's Lemma: Multivariate Version



**Theorem (Itô's Lemma for continuous semimartingales)**

Let $X = (X_t^1, \ldots, X_t^n)_{t\geq 0}$ be a continuous $\mathbb{R}^n$-valued semimartingale, and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)\,\mathrm{d}t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(t, X_t)\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t)\,\mathrm{d}[X^i, X^j]_t.$$

Special Case: $f(X, Y) = XY$: Itô product rule:

$$\mathrm{d}(XY)_t = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + \mathrm{d}[X, Y]_t$$

# Itô's Lemma: Multivariate Version

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## Theorem (Itô's Lemma for multivariate Itô processes)

Let $W$ be a $k$-dimensional standard Brownian motion, $X$ be a $\mathbb{R}^n$-valued Itô process with dynamics

$$\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}W_t$$

for sufficiently smooth functions $\mu_X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$. Let $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}$-function with gradient $\nabla f(t, X_t)$ and Hessian matrix $H_f(t, X_t)$, then

$$\mathrm{d}f(t,X_t) = \Big[\underbrace{\frac{\partial}{\partial t} f(t,X_t)}_{\in\mathbb{R}} + \underbrace{\nabla f(t,X_t)}_{\in\mathbb{R}^n} \cdot \underbrace{\mu_X}_{\in\mathbb{R}^n} + \frac{1}{2}\mathrm{tr}\Big(\underbrace{H_f(t,X_t)}_{\in\mathbb{R}^{n\times n}} \underbrace{\sigma_X}_{\in\mathbb{R}^{n\times k}} \underbrace{\sigma_X'}_{\in\mathbb{R}^{k\times n}}\Big)\Big]\mathrm{d}t$$

$$+ \underbrace{\nabla f(t,X_t)}_{\in\mathbb{R}^n} \underbrace{\sigma_X}_{\in\mathbb{R}^{n\times k}} \underbrace{\mathrm{d}W_t}_{\in\mathbb{R}^k}$$

# Example: Relative Asset Prices

# Change of Measure



## Definition (Equivalent Probability Measure)

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be *equivalent*, $\mathbb{P} \sim \mathbb{Q}$, if both measures possess the same null sets, i.e., for all events $A \in \mathcal{A}$

$$\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0.$$

- In our pricing applications, we consider equivalent probability measures that are associated to a numéraire.
- A numéraire is any self-financing portfolio $\phi$ that generates strictly positive wealth $V_t^{\phi} = \phi_t' Y_t$
- A probability measure $\mathbb{Q} \sim \mathbb{P}$ is said to be an equivalent martingale measure if for every asset with price process $Y^i$ $(i = 1, \ldots, m)$ the price expressed in terms of the numéraire $V_t^{\phi}$ is a martingale under $\mathbb{Q}$.

# Change of Measure – Radon-Nikodym Theorem



- The following theorem states how to switch between two equivalent probability measures.

**Theorem (Radon-Nikodym)**

Let $\mathbb{P} \sim \mathbb{Q}$ denote two equivalent probability measures, then there exists a unique (a.s.), positive random variable $\theta = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[\theta X], \qquad \mathbb{E}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{\theta}\right]$$

for all real-valued random variables $X$. In particular,

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[\theta\, 1_A]$$

$\theta$ is called the *Radon-Nikodym density* (or *Radon-Nikodym derivative*).

- Critical Question: How can we perform a change of measure if the market is driven by Brownian motions?

# Girsanov Theorem



**Theorem (Girsanov)**

Suppose that a measure $\mathbb{Q}$ is defined in terms of a measure $\mathbb{P}$ by the Radon-Nikodym process $(\theta_t)_{t\geq 0}$, with

$$\mathrm{d}\theta_t = -\lambda_t \theta_t \, \mathrm{d}W_t$$

where $W$ is a Brownian motion under $\mathbb{P}$ and $\lambda$ is a continuous adapted process. Then the process $\widetilde{W}$ defined by $\widetilde{W}_0 = 0$ and

$$\mathrm{d}\widetilde{W}_t = \lambda_t \, \mathrm{d}t + \mathrm{d}W_t$$

is a Brownian motion under $\mathbb{Q}$.

This works as well for vector BMs; in this case, write

$$\mathrm{d}\theta_t = -\theta_t \lambda_t' \, \mathrm{d}W_t, \quad \mathrm{d}\widetilde{W}_t = \lambda_t \, \mathrm{d}t + \mathrm{d}W_t.$$

# Some Remarks



- The stochastic differential equation $\mathrm{d}\theta_t = -\lambda_t \theta_t \,\mathrm{d}W_t$ has a unique solution, the Radon-Nikodym *process*:

$$\theta_t = \mathcal{E}(\lambda)_t = \exp\left(-\int_0^t \lambda_s \mathrm{d}W_s - \frac{1}{2}\int_0^t \lambda_s^2 \mathrm{d}s\right)$$

- The process $\mathcal{E}(\lambda)$ is called the stochastic exponential or Doléans-Dade exponential of $\lambda$.
- The Radon-Nikodym *derivative* is given by

$$\theta_T = \exp\left(-\int_0^T \lambda_s \mathrm{d}W_s - \frac{1}{2}\int_0^T \lambda_s^2 \mathrm{d}s\right)$$

- The Radon-Nikodym *process* is a $\mathbb{P}$-martingale, i.e.,

$$\theta_t = \mathbb{E}_t[\theta_T].$$