Capital Markets and Asset Pricing
Goethe Business School
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Problem Set 4

Problem 4.1 (Portfolio Selection Theory) The expected returns of two assets A and B are 8% and 12%. The volatilities are 15% and 20%. Besides, the covariance of the returns is 1.8%.

(a) Determine the expected return and volatility of a portfolio investing 30% of wealth in asset A and 70% in asset B.

Solution: $\mu_A = 0.08$, $\mu_B = 0.12$, $\sigma_A = 0.15$, $\sigma_B = 0.20$, $\sigma_{A,B} = 0.018$, $w_A = 0.3$, $w_B = 0.7$. Therefore,

$$\begin{aligned}\mu_P &= w_A\mu_A + w_B\mu_B \\ &= 0.3 \cdot 0.08 + 0.7 \cdot 0.12 \\ &= 0.108 = 10.8\%.\end{aligned}$$

The correlation between the two asset returns is $\rho_{A,B} = \frac{\sigma_{A,B}}{\sigma_A\sigma_B} = \frac{0.018}{0.15 \cdot 0.20} = 60\%$. Consequently, the portoflio variance is

$$\begin{aligned}\sigma_P^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B\rho_{A,B} \\ &= 0.3^2 \cdot 0.12^2 + 0.7^2 \cdot 0.2^2 + 2 \cdot 0.3 \cdot 0.7 \cdot 0.12 \cdot 0.2 \cdot 0.6 \\ &= 0.029185\end{aligned}$$

The portfolio volatility is $\sigma_P = \sqrt{0.029185} = 17.08\%$.

(b) Determine the portfolio weights of the minimum-variance portfolio consisting of the two assets A and B only.

Solution: We want to minimize portfolio variance

$$\sigma_P^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B\rho_{A,B}$$

where $w_B = 1 - w_A$, i.e.,

$$\sigma_P^2 = w_A^2\sigma_A^2 + (1 - w_A)^2\sigma_B^2 + 2w_A(1 - w_A)\sigma_A\sigma_B\rho_{A,B}$$

Deriving this expression w.r.t. the portfolio weight w_A yields

$$\frac{\mathrm{d}\sigma_P^2}{\mathrm{d}w_A} = 2 \cdot w_A \sigma_A^2 - 2(1 - w_A)\sigma_B^2 + 2(1 - 2w_A)\sigma_A \sigma_B \rho_{A,B} = 0$$

Solving this equation for w_A:

$$w_A = \frac{\sigma_B^2 - \sigma_A \sigma_B \rho_{A,B}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_A \sigma_B \rho_{A,B}}, \qquad w_B = 1 - w_A$$

Consequently, we obtain the following portfolio weights:

$$w_A^* = 0.83, \qquad w_B^* = 0.17$$

(c) Determine the MVP's expected return and volatility, and interpret your results

Solution: Substituting the optimal weights from (b) into the formulas form (a) yields

$$\mu_{MVP} = 8.67\%, \qquad \sigma_{MVP} = 14.74\%$$

Interpretation: Diversification is a powerful tool that allows investors to reduce risk even below the volatility of the least volatile asset, while not reducing expected returns.

$$\mu_A < \mu_{MVP} < \mu_B, \qquad \sigma_{MVP} < \sigma_A < \sigma_B$$

(d) Is there a diversification effect if the returns are not correlated?

Solution: The diversification effect is more pronounced if asset returns a less correlated. This can be seen from the portfolio variance formula

$$\sigma_P^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_A \sigma_B \rho_{A,B}.$$

Problem 4.2 (Capital Asset Pricing Model) Suppose you have estimated a stock volatility of $\sigma_i = 22\%$. Ignoring Roll's critique you proxy the market portfolio by an index, whose volatility is $\sigma_M = 16\%$. Besides you estimate a correlation between the stock return and the market return of 54%.

(a) Determine the covariance between the asset and the market and the asset's beta factor.

Solution: Covariance:

$$\sigma_{i,M} = \sigma_M \sigma_i \rho_{i,M} = 0.16 \cdot 0.22 \cdot 0.54 = 0.019.$$

Beta:

$$\beta_i = \frac{\sigma_{i,M}}{\sigma_M^2} = \frac{0.019}{0.16^2} = 0.74$$

(b) Suppose now the risk-free rate is 1.5% and you have observed a market risk premium of 6.5%. What is the expected market return? Depict the Security Market Line and explain its components.

Solution: Market risk premium is $MRP = \mu_M - r_f$, i.e., expected market return is $\mu_M = r_f + MRP = 8\%$.

The SML is given by

$$\mu_i = r_f + \beta_i(\mu_M - r_f)$$

(c) What is the expected return of the asset from (a) according to the CAPM?

Solution: Substitute the numbers into the SML:

$$\begin{aligned}\mu_i &= r_f + \beta_i(\mu_M - r_f) \\ &= 1.5\% + 0.74 \cdot 6.5\% = 6.31\%\end{aligned}$$

(d) Suppose that the CAPM holds true. What would happen if the expected return were 7.5%?

Solution: The asset's expected return would be too high according the the SML, hence its price would be too low. The market participants would recognize that the asset is undervalued and would buy this asset until its price has reached its equilibrium level and its expected return has gone down to 6.31%.

(e) Determine the asset's unsystematic risk.

Solution: Risk can be decomposed into a systematic and a idiosynchratic component:

$$\underbrace{\sigma_i^2}_{\text{total risk}} = \underbrace{\beta_i^2 \sigma_M^2}_{\text{sys. risk}} + \underbrace{\sigma_{\varepsilon_i}^2}_{\text{idios. risk}}$$

Therefore,

$$\sigma_{\varepsilon_i}^2 = \sigma_i^2 - \beta_i^2 \sigma_M^2 = 0.22^2 - 0.74^2 \cdot 0.16^2 = 0.034$$

Hence, $\sigma_{\varepsilon_i} = \sqrt{0.034} = 0.1851$. In turn, most of the volatility stems from idiosynchratic risk, which is not priced in the CAPM.

Problem 4.3 (Arbitrage Pricing Theory) We consider a two-factor APT and assume that idiosyncratic risk can be disregarded. There are two traded stocks and a risk-free asset.

(a) Explain how the model can be estimated and write down the security market line.

Solution: There are two factors f_1, f_2, which can be observed (e.g., market factor, size, inflation etc.). perform the following linear regression

$$R_{i,t} - r_f = \alpha_i + \beta_{i,1} f_{1,t} + \beta_{i,2} f_{2,t} + \varepsilon_{i,t},$$

where

- α_i is a constant for asset i.
- $f_{n,t}$ is a—potentially time-varying—systematic factor.
- $\beta_{i,n}$ is the sensitivity of the i^{th} asset to factor n (factor loading).
- $\varepsilon_{i,t}$ is the risky asset's idiosyncratic shock with mean zero.

Each systematic factor carries a risk premium $\lambda_n = \text{rp}(f_n)$. The security market line is

$$\mu_i = r_f + \beta_{i,1}\lambda_1 + \beta_{i,2}\lambda_2.$$

The risk-free rate is 1.5%. The factor loadings and expected returns of the two stocks are summarized in the following table.

Security	b_{i1}	b_{i2}	Exp. return
A	0.25	0.8	16.2%
B	0.75	0.9	21.6%

(b) Determine the risk premia of the two factors.

Solution: Substituting the numbers into the SML:

$$16.2\% = 1.5\% + 0.25\lambda_1 + 0.8\lambda_2$$
$$21.6\% = 1.5\% + 0.75\lambda_1 + 0.9\lambda_2$$

This implies $\lambda_1 = 16\%$, $\lambda_2 = 7.6\%$,

(c) You have 500 euros available. If you invest 800 euros in stock 1 and short sell 300 euros of stock 2, what are the sensitivities of your portfolio to the two factors? What is the expected return of that portfolio?

Solution: The portfolio weights are $w_A = \frac{800}{500} = 1.6$, $w_B = -\frac{300}{500} = -0.6$.

Notice that portfolio returns are linear in the factor sensitivities. Hence, the portfolio factor sensitivities are the weighted average asset sensitivities (just like expected returns), i.e.,

$$\mu_P = w_A\mu_A + w_B\mu_B$$
$$\beta_{P,n} = w_A\beta_{A,n} + w_B\beta_{B,n}$$

Substituting the weights into these formulas: $\mu_P = 12.96\%$, $\beta_{P,1} = -0.05$, $\beta_{P,2} = 0.74$.

(d) Construct a portfolio consisting of assets A and B which is insensitive to the first factor.

Solution: We need to determine W_A and $W_B = 1 - w_A$ such that $\beta_{P,1} = 0$.

$$\begin{aligned}
\beta_{P,1} &= w_A\beta_{A,1} + (1 - w_A)\beta_{B,1} \\
\beta_{P,1} - \beta_{B,1} &= w_A(\beta_{A,1} - \beta_{B,1}) \\
w_A &= \frac{\beta_{P,1} - \beta_{B,1}}{\beta_{A,1} - \beta_{B,1}} \\
&= \frac{-0.75}{0.25 - 0.75} \\
&= 1.5 \\
w_B &= -0.5
\end{aligned}$$

(e) Determine the expected return of an asset with sensitivities $\beta_{i,1} = 0.8$, $\beta_{i,2} = -0.2$.

Solution: Substituting the numbers into the SML:

$$\mu_i = 1.5\% + 0.8 \cdot 16\% - 0.2 \cdot 7.6\% = 12.78\%$$

Problem 4.4 (Fama-French Model and Carbon Premium) Consider the following Fama-French-three-factor-model.

Factor	Market	Size	Book
Premium	4.8%	3.6%	5.2%

(a) Suppose there is an asset for which all factor sensitivities are positive. Characterize this asset.

Solution: Small (positive sensitivity to the size factor) value (positive sensitivity to the BTM factor) stock (positively related to the market factor), or a long derivative (e.g., call option) on such a stock.

(b) Consider an asset where the sensitivities are $\beta_{i,M} = 0.9$, $\beta_{i,size} = -0.5$, $\beta_{i,book} = 0.6$. Predict its expected return if the risk-free rate is 1%.

Solution: The SML reads

$$\begin{aligned}\mu_i &= r_f + \sum_{n=1}^{3} \beta_{i,n}\lambda_n \\ &= 1\% + 0.9 \cdot 4.8\% - 0.5 \cdot 3.6\% + 0.6 \cdot 5.2\% \\ &= 6.64\%.\end{aligned}$$

(c) Suppose now there is an asset whose sensitivities are $\beta_{i,M} = 0.8$, $\beta_{i,size} = 0.5$. The expected rate of return is 9.2%. Determine its sensitivity to the BTM factor.
Solution: The SML reads

$$\mu_i = r_f + \sum_{n=1}^{3} \beta_{i,n}\lambda_n,$$

which must be solved for $\beta_{i,book}$. Therefore,

$$\begin{aligned}\beta_{i,book} &= \frac{\mu_i - r_f - \beta_{i,M}\lambda_M - \beta_{i,size}\lambda_{size}}{\lambda_{book}} \\ &= \frac{9.2\% - 1\% - 0.8 \cdot 4.8\% - 0.5 \cdot 3.6\%}{5.2\%} \\ &= 0.49\end{aligned}$$

Following Bolton & Kaspercyk (2021), we now consider an extension of the Fama-French model where carbon intensity is added as a factor to the model. Bolton & Kaspercyk (2021) find that there exist a *carbon premium*, i.e., more carbon-intensive industries have higher expected returns.

(d) Explain why there might be a carbon premium.

Solution: There is a systematic risk factor in the market, which refers to transition risk. Carbon-intensive industries are more likely to be negatively affected from political measures that aim at reducing carbon dioxide. For some industries (e.g., coal power plants, oil companies) these risks could even be existence-threatening. Investors want to be compensated for these risks and thus require a carbon premium.

(e) Suppose we regress returns against the Fama-French factors and the logarithm of total emissions generated by the firm.

$$R_{i,t} = \alpha_i + \beta_{i,market} \cdot (R_{m,t} - r_f) + \beta_{i,size} \cdot SMB_t + \beta_{i,book} \cdot HML_t + \beta_{i,carbon} \log(E_{i,t}) + \varepsilon_{i,t},$$

where the emission factor carries a risk premium $\lambda_{carbon} = 1.5\%$. Suppose $\beta_{i,carbon} = 0.3$ (the other sensitivities might be affected by adding a new factor to the model). What does this mean in terms of expected returns, risk, and volatility?

Solution: The firm-specific carbon premium is $0.3 \cdot 1.5\% = 0.45\%$. This is because of the significant systematic carbon risk, for which investors want to be compensated. Notice that higher systematic risk does not necessarily imply higher volatility.