Capital Markets and Asset Pricing
Goethe Business School
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Problem Set 3

Problem 3.1 (State Pricing and Default Risk) Consider a firm that is financed by equity and debt. The current stock price is $S_0 = 50$. This firm has emitted two types of bonds with notional $N = 100$ each: A senior bond currently trading at 95 and a junior bond currently trading at 89. Assume that the firm faces significantly default risk. After one year, the following three scenarios can occur:

– Scenario 1: No default has occurred, and the stock price has increased by 20%.

– Scenario 2: The firm has gone bankrupt and the LGD of the junior bond is 40%.

– Scenario 3: The firm has gone bankrupt and the LGD of the senior bond is 40%, while the junior bond has been wiped out.

(a) Visualize this one-period state pricing model by a tree diagram.

Solution:

$$
\begin{tikzcd}
 & \begin{pmatrix}60\\100\\100\end{pmatrix} \\
\begin{pmatrix}50\\95\\89\end{pmatrix} \arrow[ur] \arrow[r] \arrow[dr] & \begin{pmatrix}0\\100\\60\end{pmatrix} \\
 & \begin{pmatrix}0\\60\\0\end{pmatrix}
\end{tikzcd}
$$

(b) Determine the prices of the elementary securities and check whether the market is free of arbitrage. What would happen if the junior bond were trading at 81?

Solution: The prices of the elementary securities satisfy the following linear system:

$$
\begin{aligned}
50 &= 60\pi_u \\
95 &= 100\pi_u + 100\pi_m + 60\pi_d \\
89 &= 100\pi_u + 60\pi_m
\end{aligned}
$$

Solving this linear system yields $\pi_u = 0.8333$, $\pi_m = 0.0944$, $\pi_d = 0.0278$. Since all prices are positive, the market is free of arbitrage.

If the junior bond were trading at 81, we would have calculated $\pi_u = 0.8333$, $\pi_m = -0.5556$, $\pi_d = 0.2153$. Then, the market would contain arbitrage opportunities.

(c) Determine the risk-free rate in this economy for both discretely compounded and continuously compounded interest rates.

Solution: The price of the risk-free asset is

$$B_0 = \sum_s \pi_s = 0.8333 + 0.0944 + 0.0278 = 0.9556.$$

Hence, the risk-free rate $r_{discrete} = \frac{1}{B_0} - 1 = 4.65\%$. The continuously compounded rate relates to the discretely compounded rate via

$$1 + r_{discrete} = \mathrm{e}^{r_{cont}}$$

Therefore, $r_{cont} = \log(1 + r_{discrete}) = \log(1.0465) = 4.55\%$

(d) Determine the risk-neutral survival probabilities of both bonds.

Solution: The risk-neutral probabilities are given by $q_s = (1 + r_{discrete}) \cdot \pi_s$. We obtain:

$$q_u = 87.21\%, \qquad q_m = 9.88\%, \qquad q_d = 2.91\%$$

The junior bond defaults in states m and d, and the senior bond defaults in state d only. Therefore, the default probabilities and survival probabilities are given by

$$
\begin{aligned}
\mathsf{PD}^{\mathbb{Q}}_{junior} &= q_m + q_d = 12.79\%, \qquad & \mathsf{SP}^{\mathbb{Q}}_{junior} &= 1 - \mathsf{PD}^{\mathbb{Q}}_{junior} = 87.21\% \\
\mathsf{PD}^{\mathbb{Q}}_{senior} &= 2.91\%, & \mathsf{SP}^{\mathbb{Q}}_{senior} &= 1 - \mathsf{PD}^{\mathbb{Q}}_{senior} = 97.09\%.
\end{aligned}
$$

(e) Determine the price of a put option on the senior bond with a strike price of $K = 100$, and explain why it hedges the senior bond's default risk.

Solution: This put option provides protection against a default of the senior bond. The put is worthless if the bond is fully paid back at time $T = 1$ (scenarios u and m). In scenario d, the senior bond defaults, but the holder of the put option can sell it at par (instead of 80), i.e, she gains 20. Therefore, the holder of this put option has hedged her default risk. In this sense, the put option is similar to a CDS.

Price of the put option:

$$P_0 = \frac{1}{1+r} q_d P_d = \frac{2.91\% \cdot 40}{1.0465} = 1.12$$

Problem 3.2 (Merton's Firm Value Model) Consider a firm with a firm value of 100,000,000 Euro. The asset volatility is $\sigma = 0.1$. The firm is financed by equity and a zero bond with notional 90,000,000 and maturity at $T = 10$. The risk-free rate is $r = 3.5\%$.

(a) Determine the market price of equity and debt as well as the firm's leverage.

Solution: The Black-Scholes formula yields:

$$\begin{aligned}
E_0 &= V_0 \Phi(d_1) - F e^{-rT} \Phi(d_2) \\
D_0 &= V_0 - E_0 = F e^{-y^d(T)T} \\
d_1 &= \frac{\ln(V_0/F) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}$$

Substituting the numbers into this formula yields $E_0 = 37.416$, $D_0 = 62.584$. The firm's leverage is thus given by $\ell = \frac{D_0}{E_0} = 1.6727$.

(b) Calculate the credit spread, i.e., the yield spread between the corporate bond and the risk-free rate.

Solution: $D_0 = 62.584$. The yield on the treasury bond is obviously given by the risk-free rate. The yield y_D on the defaultable corporate bond is determined via the condition

$$\begin{aligned}
D_0 &\overset{!}{=} F \cdot e^{-y_D \cdot T} \\
\Rightarrow y_D &= -\frac{1}{T} \ln\left(\frac{D_0}{F}\right),
\end{aligned}$$

so that the yield spread is given as

$$y_D - r = -\frac{1}{T} \ln\left(\frac{D_0}{F}\right) - r.$$

In the example we obtain

$$\begin{aligned} y_D &= -\frac{1}{10}\ln\left(\frac{64.656}{90}\right) \\ &= 0.0363 \end{aligned}$$

and thus $y_D - r = 0.0013 = 0.13\%$.

Problem 3.3 (Credit and Liquidity Premium) Suppose that a US-firm has issued two types of bonds L, I that are traded on two different markets. While both bonds are exposed to credit risk, they differ in their liquidity. The liquid bonds (L) can be traded without bid-ask spread, while the illiquid bonds (I) are facing a bid-ask-spread and a liquidity premium. The following table summarizes the data.

T	$P_{0,L}$	$P_{0,I}$	c	N
1	101.15	100	1.0%	100
2	98.00	96.7	2.0%	100

The reported prices $P_{0,L}$, $P_{0,I}$ are the current mid-prices of the liquid and illiquid bonds, respectively. The illiquid bonds are currently trading at a bid-ask-spread of 0.3%.

(a) Determine the bid price and the ask price of the illiquid bonds.

Solution:

$$\text{Bid-Ask Spread} = \frac{\text{Ask Price} - \text{Bid Price}}{\text{Mid Price}}.$$

Therefore, the absolute spread is Ask Price − Bid Price $= 0.3\% \cdot$ Mid Price.

1st bond: Ask Price − Bid Price $= 0.3\% \cdot 100 = 0.3$. Since the mid price is the average of bid and ask prices, we obtain:

$$\text{Ask Price} = 100 + 0.5 \cdot 0.3 = 100.15$$

$$\text{Bid Price} = 100 - 0.5 \cdot 0.3 = 99.85$$

2nd bond: Ask Price − Bid Price $= 0.29\% \cdot 96.7 = 0.29$.

$$\text{Ask Price} = 96.7 + 0.5 \cdot 0.29 = 96.845$$

$$\text{Bid Price} = 96.7 - 0.5 \cdot 0.29 = 96.555$$

(b) Determine the liquidity premium of the illiquid bonds.

Solution: The first step is to determine the YTM for all bonds, see Problem 1.4 for the liquid bonds: $y_L(1) = -0.15\%$, $y_L(2) = 3\%$.

Performing exactly the same calculations for the illiquid bonds yields $y_I(1) = 1\%$, $y_I(2) = 3.63\%$.

The yield spread between the liquid bonds and the illiquid bonds are the yield spread, i.e.,

1 year: liquidity premium $= y_I(1) - y_L(1) = 1.15\%$.

2 years: liquidity premium $= y_I(2) - y_L(2) = 0.63\%$.

(c) Suppose that the yield curve of the corresponding US Government bonds (same notional, payment dates, and coupon rates as the corporate bonds) is given by $y(1) = -0.25\%$, $y(2) = 2.5\%$. Determine the credit spread of all bonds.

Solution:

$$\text{Total Spread} = \text{Credit Spread} + \text{Liquidity Premium}$$

Therefore,

$$\text{Credit Spread} = \underbrace{\text{Total Spread}}_{y_x(T)-y(T)} - \underbrace{\text{Liquidity Premium}}_{y_x(T)-y_L(T)}$$

The credit spread is obviously identical for the liquid and illiquid bonds. For the liquid bonds ($x = L$), the liquidity premium is zero, hence

$$\text{Credit Spread}(1) = y_L(1) - y(1) = 0.1\%$$
$$\text{Credit Spread}(2) = y_L(2) - y(2) = 0.5\%$$