Capital Markets and Asset Pricing
Goethe Business School
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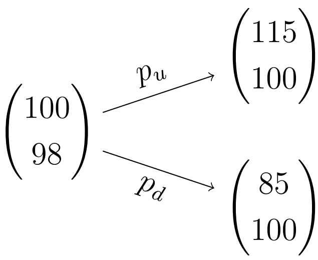
Problem Set 2

Problem 2.1 (State Pricing) Consider the following state pricing model with two assets (stock and risk-free asset) and two states:



(a) Calculate the prices of the elementary securities in this model.

Solution: We build the linear system

$$100 = 115\pi_u + 85\pi_d$$
$$98 = 100\pi_u + 100\pi_d$$

Solving this system yields $\pi_d = 0.4233$, $\pi_u = 0.5567$.

(b) Is this market free of arbitrage? Is the market complete? Explain your answer.

Solution: This market is obviously free of arbitrage, since $\pi_s > 0$ for all states s. It is complete since the prices of the elementary securities are unique.

(c) Determine the risk-free rate and the risk-neutral probabilities.

Solution: The risk-free rate is given by

$$r_f = \frac{100}{98} - 1 = 2.04\%$$

The risk-neutral probabilities are given by $\pi_s(1 + r_f)$, i.e.,

$$q_d = 0.4233 \cdot 1.0204 = 43.20\%$$
$$q_u = 0.5567 \cdot 1.0204 = 56.80\%.$$

(d) Determine the price of a European put option on the stock with a strike price of 100.

Solution: The put option profile is $P_T = \max(K - S_T; 0)$. In the up state we get $P_u = \max(100 - 115; 0) = 0$ (option will not be exercised), and in the down state, we get $P_d = \max(100 - 85; 0) = 15$. Then, the option price is

$$\begin{aligned} P_0 &= \frac{1}{1+r}(q_u P_u + q_d P_d) \\ &= \frac{1}{1.0204} \cdot 0.4320 \cdot 15 \\ &= 6.35 \end{aligned}$$

(e) Provide the replication strategy for the put option from part (d).

Solution: The replication strategy satisfies the following linear system, where φ_S denotes the number of stocks held, and φ_B the position in the risk-free assets, respectively:

$$\begin{aligned} 15 &= 85\varphi_S + 100\varphi_B \\ 0 &= 115\varphi_S + 100\varphi_B \end{aligned}$$

Therefore, $\varphi_S = -0.5$ (short-selling), $\varphi_B = 0.575$.

The price of the put option can thus be calculated as

$$\begin{aligned} P_0 &= S_0\varphi_S + B_0\varphi_B \\ &= 100 \cdot (-0.5) + 98 \cdot 0.575 \\ &= 6.35 \end{aligned}$$

(f) Determine the expected stock return under the risk-neutral probability measure. What do you observe?

Solution: The stock return in the two states are $R_s = \frac{S_s}{S_0} - 1$, hence $R_u = \frac{115}{100} - 1 = 15\%$, $R_d = \frac{85}{100} - 1 = -15\%$.

Expected return is

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[R_s] &= q_u R_u + q_d R_d \\ &= 0.568 \cdot (-15\%) + 0.432 \cdot 15\% \\ &= 2.04\% = r_f \end{aligned}$$

This result holds for all traded assets (and in all arbitrage-free models).

Problem 2.2 (Option Pricing in a Two-Period Model) Consider a two-period binomial tree with two assets (stock and risk-free asset). Today's stock price is 100. In each period, the stock price can either increase by 12% or decrease by 12%. A period corresponds to six months. The TSIR is flat and the annual interest rate is 4.04%.

(a) Determine the spot rate over one period (= 6 months).

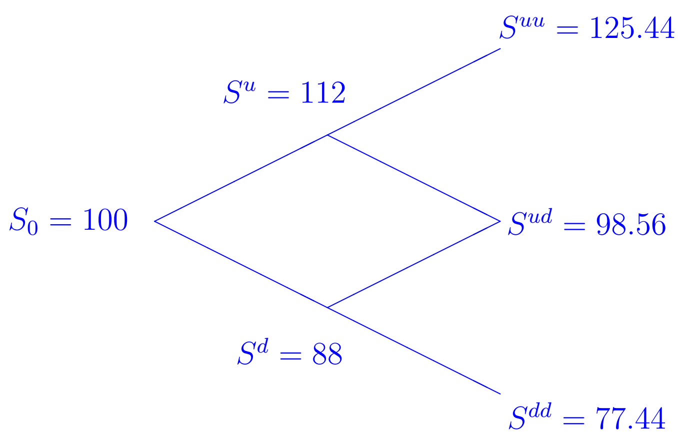
Solution:

$$(1 + r(0.5) \cdot 0.5)^2 = 1 + r(1)$$

Therefore, $r(0.5) = 2(\sqrt{1.0404} - 1) = 4\%$ (annual rate). Consequently, the one-period rate is $r = 2\%$.

(b) Set up the binomial tree for the stock price.

Solution:



(c) Calculate the risk-neutral probabilities.

Solution: Of course, we can determine the risk-neutral probabilities by our usual three step procedure (i) determine state prices π_s; (ii) determine the risk-free rate for one period r; (iii) Calculate the risk-neutral probabilities $q_s = \pi_s(1 + r)$.

However, there is a neat formula for the risk-neutral probabilities, which is given by $q_u = \frac{u-r}{u-d}$, where u denotes the return in the up-state and d the return in the down state. Therefore:

$$
\begin{aligned}
q_u &= \frac{r-d}{u-d} = \frac{2\% - (-12\%)}{12\% - (-12\%)} = 58.33\% \\
q_u &= 1 - q_d = 41.67\%
\end{aligned}
$$

(d) Determine the price of a European call option with a strike price of 90.

Solution: The payoff of the call option is $C_T = \max(S_T - K; 0)$, i.e.,

$$
\begin{aligned}
C_{uu} &= \max(125.44 - 90; 0) = 34.44 \\
C_{du} = C_{ud} &= \max(98.56 - 90; 0) = 8.56 \\
C_{dd} &= \max(77.44 - 90; 0) = 0
\end{aligned}
$$

Consequently, the price of the option is

$$
\begin{aligned}
C_0 &= \frac{1}{(1+r)^2}\left[q_u^2 C^{uu} + 2 q_u q_d C^{ud} + q_d^2 C^{dd}\right] \\
&= \frac{1}{1.02^2}\left[(58.33\%)^2 \cdot 34.44 + 2(41.67\%)(58.33\%) \cdot 8.56\right] \\
&= 15.26
\end{aligned}
$$

Problem 2.3 (Black-Scholes Model) Consider a Black-Scholes model with the following parameters: the risk-free rate is $r = 0.5\%$, the volatility is $\sigma = 22\%$, and the current stock price is $S_0 = 95$.

(a) Determine the price of a European call option with strike price $K = 100$ and maturity at $T = 2$.

Solution: The Black-Scholes formula for a call option reads

$$
C_0 = S_0 \Phi(d_1) - K \cdot \mathrm{e}^{-rT} \Phi(d_2), \qquad P_0 = C_0 - S_0 + K \cdot \mathrm{e}^{-rT}
$$

with

$$
d_1 = \frac{\ln(S_0/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.
$$

Substituting the numbers into this formula yields

$$d_1 = 0.023, \qquad d_2 = -0.288.$$

Plugging these numbers into the cumulative distribution function of the standard normal distribution (Excel: "=NORM.S.DIST(D1;True)", or using a CDF table), we obtain

$$\Phi(d_1) = 0.509, \qquad \Phi(d_2) = 0.387.$$

Therefore, the Call price equals

$$C_0 = 95 \cdot 0.509 - 100 \cdot \mathrm{e}^{-0.005 \cdot 2} \cdot 0.387 = 10.09.$$

(b) Write down the replication portfolio for the option from part (a).

Solution: The replication portfolio is $\varphi_S = \Phi(d_1) = 0.509$, $\varphi_B = -\Phi(d_2)K = -38.7$

(c) Suppose you want to price a reverse convertible bond with $N = 100$, $k = 1$ paying coupons at a rate of $c = 2\%$ and maturing in 2 years. Suppose the term structure of interest rates is flat and there is no credit risk.

Solution: We split up the problem into two parts:

1st step: price the ordinary bond, i.e., determine the present value of the bond payments ignoring the option profile: $PV = 2\mathrm{e}^{-0.005} + 102e^{-2\dot{0}.005} = 102.98$.

2nd step: seller of the reverse convertible bond has the right (but not the obligation) to pay back the notional N or to deliver a number k of stocks, i.e., $k \cdot S_T$. Therefore, the holder of the reverse convertible bond is short in the put option with terminal payoff $P_T = \max(N - k \cdot S_t; 0)$. In this problem $k = 1$, i.e., a put option on this stock with strike price $N = 100$. According to Problem (a) we can use the put-call parity (identical strike price, stock price, interest rate, volatility, and time-to-maturity):

$$\begin{aligned} P_0 &= C_0 - S_0 + K \cdot \mathrm{e}^{-r \cdot T} \\ &= 10.09 - 95 + 100\mathrm{e}^{-0.005 \cdot 2} \\ &= 14.10. \end{aligned}$$

Hence, the price of the reverse convertible is $102.98 - 14.10 = 88.88$.

By contrast, the price of the corresponding convertible bond is $102.98 + 10.09 = 113.07$ (long call).

(d) Explain why the drift rate of the stock price μ does not show up in the Black-Scholes formula.

Solution: The Black-Scholes model exploits risk-neutral pricing, and under $\mathbb{Q}$ all traded assets grow at the risk-free rate.