Capital Markets and Asset Pricing
Goethe Business School
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Problem Set 1

Problem 1.1 (Bond Pricing) You have a bond with notional $N = 1,000$ which matures in 240 days (30/360 usance). The coupon is 6% and the clean price is 990.

(a) Calculate the accrued interest and the dirty price. Explain the difference between clean and dirty bond prices.

Solution: The last coupon payment was 120 days ago. Hence, accrued interests are given by

$$\text{Accrued interests} = C \cdot \frac{d}{B} = 60 \cdot \frac{120}{360} = 20.$$

Therefore, $P_0 = 990 + 20 = 1,010$.

(b) Assume capital market usance and redo problem (a).

Solution: Capital market usance is act/act. Therefore, the last coupon payment was 125 days ago.

$$\text{Accrued Interests} = C \cdot \frac{d}{B} = 60 \cdot \frac{125}{365} = 20.55.$$

Therefore, $P_0 = 990 + 20.55 = 1,010.55$.

Problem 1.2 (Discretely compounded vs. Continuously) Current prices of zero-coupon bonds ($N = 100$) with maturities in 6 months, and twelve months are $P_0(T = 0.5) = 99.8$, and $P_0(T = 1) = 98.4$

(a) Determine the discretely compounded spot rates $r(0.5)$, $r(1)$, and the forward rate for an investment over the second period $f(0.5, 1)$.

Solution: We are looking for discretely compounded interest rates for a time horizon below one year. The spot rates are thus given by

$$P_0(T)\big(1 + r(T)T\big) = N \qquad \Longleftrightarrow \qquad r(T) = \frac{1}{T}\Big(\frac{N}{P_0(T)} - 1\Big)$$

This yields $r(0.5) = 0.40\%$, $r(1) = 1.63\%$.

The no arbitrage condition for the forward rate is

$$\big(1 + r(0.5) \cdot 0.5\big)\big(1 + f(0.5,1) \cdot 0.5\big) = \big(1 + r(1) \cdot 1\big)$$

The forward rate for the second period is thus $f(0.5, 1) = 2.85\%$.

(b) Determine the corresponding continuously compounded interest rates.

Solution: The relation between discretely compounded and continuously compounded rates (for time horizons below 1 year) is

$$\mathrm{e}^{r_c \cdot T} = 1 + r_d \cdot T \qquad \Longleftrightarrow \qquad r_c = \frac{1}{T}\log(1 + r_d \cdot T)$$

Therefore,

$$\begin{aligned} r_c(1/2) &= 0.40\% \\ r_c(1) &= 1.61\% \\ f_c(0.5,1) &= 2.83\% \end{aligned}$$

Problem 1.3 (TSIR and Bond Pricing) Suppose the following bond data of German Bundesanleihen is given

	T	P_0	c	N
1st Bond	1	101.15	1.0%	100
2nd Bond	2	98.00	2.0%	100
3rd Bond	3	99.50	1.5%	100

(a) Back out the spot and forward rates from this bond data. What kind of term structure do you get? What do you observe?

Solution: We use the bootstrapping technique:

1st Bond: $101.15 = 101 \cdot \mathrm{e}^{-r(1)}$. Therefore,

$$r(1) = -\log\left(\frac{101.15}{101}\right) = -0.00148 = -0.15\%.$$

2nd bond: $98 = 2 \cdot \mathrm{e}^{-r(1)} + 102 \cdot \mathrm{e}^{-2r(2)}$. Therefore,

$$r(2) = -\frac{1}{2}\log\left(\frac{98 - 2 \cdot \mathrm{e}^{0.00148}}{102}\right) = 3.03\%$$

3rd bond: $99.5 = 1.5 \cdot \mathrm{e}^{-r(1)} + 1.5 \cdot \mathrm{e}^{-2r(2)} + 101.5 \cdot \mathrm{e}^{-3r(2)}$. Therefore,

$$r(3) = -\frac{1}{3}\log\left(\frac{99.5 - 1.5 \cdot \mathrm{e}^{0.00148} - 1.5 \cdot \mathrm{e}^{-2 \cdot 0.0303}}{101.5}\right) = 1.65\%$$

The term structure is hump-shaped.

Forward rates:

$$\begin{aligned} f(0) &= r(1) = -0.15\% \\ f(1) &= 2r(2) - r(1) = 6.21\% \\ f(2) &= 3r(3) - r(1) - r(2) = 2.07\% \end{aligned}$$

Observation: term structure of forward rates is steeper than term structure of spot rates.

(b) Determine the current yield, simple redemption yield, and yield-to-maturity for the second bond. Explain the differences between these yield concepts.

Solution: $P_0 = 98$, $N = 100$, $C = 2$.

$$\begin{aligned} y_{current} &= \frac{C}{P_0} = \frac{2}{98} = 2.04\% \\ y_{simple} &= \frac{C}{P_0} + \frac{1}{T}\frac{N - P_0}{P_0} = 2.04\% + \frac{1}{2} \cdot \frac{100 - 98}{98} = 3.06\% \end{aligned}$$

The yield-to-maturity is given by the solution to the equation

$$\begin{aligned} P_0 &= \sum_{t=1}^{T} C\mathrm{e}^{-yt} + N\mathrm{e}^{-yT} \\ 98 &= 2e^{-y} + 102e^{-2y} \end{aligned}$$

To solve this equation, we substitute $x = e^{-y}$, $x^2 = e^{-2y}$. This leads to the following quadratic equation:

$$0 = -98 + 2x + 102x^2$$

Therefore,

$$\begin{aligned} x_{1/2} &= \frac{-2 \pm \sqrt{2^2 + 4 \cdot 102 \cdot 98}}{2 \cdot 102} \\ x_1 &= 0.9704, \qquad x_2 = -0.9900 \end{aligned}$$

Resubstitute $x = e^{-y}$, i.e., $y = -\log(x)$. Therefore, x_2 does not deliver a real solution. YTM is given by $y = -\log(0.9704) = 3\%$.

(c) Suppose there is another German Bundesanleihe with a coupon rate of 3% maturing in 3 years. Determine its arbitrage-free price.

Solution:

$$P_0 = 3 \cdot \mathrm{e}^{0.00148} + 3 \cdot \mathrm{e}^{-2 \cdot 0.0303} + 103 \cdot \mathrm{e}^{-3 \cdot 0.0165} = 103.85$$

(d) What would you do if this bond were trading at 102?

Solution: This bond would be undervalued. There is an arbitrage opportunity: Buy the bond at a price of 102 and sell the replication portfolio, whose price is 103.85. Then, you make an arbitrage profit of 1.85.

(e) What would be different if the bonds were US Treasuries? Write down the pricing equations using spot rates.

Solution: US Treasuries pay coupons semi-anually. The pricing equation for a bond with maturity T (expressed in years) is

$$P_0 = \sum_{t=1}^{2T} \mathsf{e}^{-r(t/2)\frac{t}{2}} C + \mathsf{e}^{-r(T)T} N$$

Here, we cannot back out the spot rates since we do not observe bond prices for $T \in \{0.5, 1.5, 2\}$

Problem 1.4 (Interest Rate Exposure) We consider a coupon bond with a coupon rate of 2% per annum, a notional of 100 euros, and a time to maturity of 3 years. Coupons are paid annually. Its continuously compounded yield-to-maturity is 1%.

(a) Determine the bond's price P_0, duration D, and convexity Γ.

Solution:

$$P_0 = \sum_{t=1}^{T} C\mathsf{e}^{-yt} + N\mathsf{e}^{-yT}$$
$$P_0 = 2e^{-0.01} + 2e^{-2 \cdot 0.01} + 102e^{-3 \cdot 0.01}$$
$$P_0 = 102.93$$

$$D = \frac{\sum_{t=1}^{T} t \cdot C\mathrm{e}^{-yt} + T\mathrm{e}^{-yT}}{P_0}$$

$$D = \frac{1 \cdot 2e^{-0.01} + 2 \cdot 2e^{-2 \cdot 0.01} + 3 \cdot 102e^{-3 \cdot 0.01}}{102.93}$$

$$D = 2.9425$$

$$\Gamma = \frac{\sum_{t=1}^{T} t^2 \cdot C\mathrm{e}^{-yt} + T^2\mathrm{e}^{-yT}}{P_0}$$

$$\Gamma = \frac{1^2 \cdot 2e^{-0.01} + 2^2 \cdot 2e^{-2 \cdot 0.01} + 3^2 \cdot 102e^{-3 \cdot 0.01}}{102.93}$$

$$\Gamma = 8.7509$$

(b) What change in the yield-to-maturity does the duration predict if the price goes down by $1?

Solution:

$$\Delta P \approx -D \cdot P_0 \cdot \Delta y$$

$$\Delta y \approx -\frac{\Delta P}{D \cdot P_0}$$

$$\Delta y \approx -\frac{-1}{2.9425 \cdot 102.93} = \frac{1}{2.9425 \cdot 102.93}$$

$$\Delta y \approx 0.0033 = 0.33\%$$

(c) Suppose you observe rising interest rate and the yield-to-maturity goes down by 80bps=0.8%. Predict the price change using (i) duration only, (ii) duration and convexity, and compare it to (iii) the true price change.

Solution: (i) Duration only, $\Delta y = -0.008$:

$$\Delta P \approx -D \cdot P_0 \cdot \Delta y$$

$$\Delta P \approx -2.9425 \cdot 102.93 \cdot (-0.008)$$

$$\Delta P \approx 2.42$$

(ii) Duration and convexity:

$$\Delta P \approx -D \cdot P_0 \cdot \Delta y + \frac{1}{2}\Gamma \cdot P_0 \cdot \Delta y$$

$$\Delta P \approx 2.42 + 0.5 \cdot 8.7509 \cdot 102.93 \cdot (-0.008)^2$$

$$\Delta P \approx 2.45$$

(iii) True change: $y_{new} = y_{old} + \Delta y = 0.002$

$$P_0 = \sum_{t=1}^{T} C\mathsf{e}^{-y_{new}t} + N\mathsf{e}^{-y_{new}T}$$

$$P_0 = 105.38$$

Therefore, $\Delta P = 105.38 - 102.93 = 2.45$

(d) Suppose you hold an equally weighted portfolio of the bond above and a zero-coupon bond with a maturity in 5 years and a current price of 89. Determine the portfolio duration and convexity.

Solution: For the zero bond, we have $D = T$, $\Gamma = T^2$, i.e.,

$$D_1 = 5, \qquad \Gamma_1 = 25, \qquad D_2 = 2.94, \qquad \Gamma_2 = 8.75$$

The portfolio is equally weighted, i.e., $w_1 = w_2 = 0.5$. Therefore,

$$D_P = 0.5 \cdot 5 + 0.5 \cdot 2.94 = 3.97$$

$$\Gamma_P = 0.5 \cdot 25 + 0.5 \cdot 8.75 = 16.87$$