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Timing of uncertainty resolution

- An agent with additive utility is indifferent between early or late resolution of uncertainty.
- Consider two consumption streams

- 1 In each period $t = 0, 1, \dots, T$, consumption is i.i.d. with

$$\mathbb{P}(C_t = \bar{C}) = \mathbb{P}(C_t = \underline{C}) = 0.5.$$

where $\bar{C} > \underline{C}$.

- 2 In each period $t = 1, 2, \dots, T$, $C'_t = C_0$ where

$$\mathbb{P}(C'_0 = \bar{C}) = \mathbb{P}(C'_0 = \underline{C}) = 0.5.$$

- With additive utility, both streams generate the same indirect utility (check!).
- If you prefer one of them, you cannot have time-additive utility!

Intertemporal Substitution vs. Risk Aversion

- Agents typically dislike fluctuations in their consumption streams over time
- Suppose $C = \frac{1}{2}(\bar{C} + \underline{C})$. Consider three consumption streams
 - ① Consumption is **constant** $C_t = C$ for all $t = 0, 1, \dots, T$
 - ② Consumption varies **over time** $C'_t = \bar{C}$ if $t = 0 \bmod 2$ and $C'_t = \underline{C}$ if $t = 1 \bmod 2$
 - ③ Consumption varies **across states** In each period $t = 1, 2, \dots, T$, consumption is i.i.d.

$$\mathbb{P}(C_t = \bar{C}) = \mathbb{P}(C_t = \underline{C}) = 0.5.$$

- Agents (typically) prefer C over C' due to their aversion against intertemporal variation.
- Agents (typically) prefer C over C'' due to their aversion against variation across states (risk).

Intertemporal Substitution vs. Risk Aversion

- For time additive utility, both is determined by the concavity of the utility function, e.g., CRRA-utility: $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$.
 - Relative risk aversion is given by

$$RRA = -\frac{Cu''(C)}{u'(C)} = \gamma$$

- Elasticity of intertemporal substitution measures the responsiveness of the growth rate of consumption to the real interest rate (Hall 1988).

$$EIS = \frac{d\Delta c_{t+1}}{dr_t^f} = \dots = \frac{1}{\gamma}$$

- **Substitution Effect:** If r^f goes up, the agent might reduce consumption and saves more to increase future consumption.
- **Wealth Effect:** If r^f goes up, the agent might feel wealthier and consumes more.
- Both properties are inseparably tied together.

Elasticity of Intertemporal Substitution

$$EIS = \frac{d\Delta C_{t+n}}{dr_t^f}$$

$$r_t^f = \delta + \gamma \mu_c - \frac{1}{2} \gamma^2 \Delta c^2$$

$$= \delta + \gamma (\Delta C_{t+n} - \Delta c v_{t+n}) - \frac{1}{2} \gamma^2 \Delta c^2$$

$$\Rightarrow \Delta C_{t+n} = \frac{r_t^f - \delta + \frac{1}{2} \gamma^2 \Delta c^2}{\gamma} + \Delta c v_{t+n}$$

$$\frac{d\Delta C_{t+n}}{dr_t^f} = \frac{1}{\gamma} = \boxed{EIS = \frac{1}{RDA}}$$

Elasticity of Intertemporal Substitution

$$r_{t+1}^I = \delta + \frac{1}{\psi} r^c - \frac{1}{2} \gamma^2 \Delta c^2$$

$$\text{EIS} = \psi$$

Example

① $\text{EIS} > 1$, e.g., $\psi = 2$

$\Rightarrow r_{t+1}^I$ increases by 1%.

$$\Rightarrow \frac{d\Delta c_{t+1}}{dr_{t+1}^I} = 2 \Rightarrow d\Delta c_{t+1} = 2 \cdot \overbrace{dr_{t+1}^I}^{1\%} = 2\%$$

Elasticity of Intertemporal Substitution

Here, the substitution effect dominates the wealth effect.

$$\textcircled{2} \quad EIS < 1, \quad \psi = \frac{1}{2}$$

r_t^d increases by 1%

ΔC_{t+1} increases by just 0.5%

\Rightarrow wealth effect dominates the substitution effect.

Recursive Utility

- Recursive Utility is one possible way of addressing some of the previous issues.
- A **recursive utility index** \mathcal{U} can be expressed as

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = W_t(C_t, \mathcal{U}_{t+1}(C_{t+1}, C_{t+2}, \dots))$$

where W is an **intertemporal aggregator**.

- W describes the aggregation of present consumption and future utility.
- The aggregator takes utility from current consumption C_t and expected utility from future consumption \mathcal{U}_{t+1} into account.

Example: Time-Additive Utility

- Choose the linear aggregator

$$U_t(C_t, C_{t+1}, \dots) = u(C_t) + e^{-\delta} \mathbb{E}_t[U_{t+1}(C_{t+1}, C_{t+2}, \dots)]$$

- Then, time- t utility is given by

$$\begin{aligned} U_t &= u(C_t) + e^{-\delta} \mathbb{E}_t[u(C_{t+1}) + e^{-\delta} \mathbb{E}_t[U_{t+2}]] \\ &= u(C_t) + e^{-\delta} \mathbb{E}_t[u(C_{t+1})] + e^{-2\delta} \mathbb{E}_t[u(C_{t+2}) + e^{-\delta} \mathbb{E}_t[U_{t+3}]] \\ &= u(C_t) + e^{-\delta} \mathbb{E}_t[u(C_{t+1})] + e^{-2\delta} \mathbb{E}_t[u(C_{t+2})] + e^{-3\delta} \mathbb{E}_t[U_{t+3}] \\ &= \dots \\ &= \sum_{k=0}^T e^{-\delta k} \mathbb{E}_t[u(C_{t+k})] \end{aligned}$$

- Standard time-additive utility is a special case of recursive utility for a linear aggregator.

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- Consider the following CES aggregator

$$U_t(C_t, C_{t+1}, \dots) = \left[\alpha C_t^{1-\phi} + \beta \text{CE}_t(U_{t+1})^{1-\phi} \right]^{\frac{1}{1-\phi}}$$

where $\phi > 0$ and

$$\text{CE}_t(U_{t+1}) = G^{-1}(\mathbb{E}_t[G(U_{t+1})])$$

for increasing and concave functions G .

- The more concave G is, and the more uncertain the consumption stream is, the lower is the certainty equivalent.
- Most of the literature assumes $G(x) = \frac{1}{1-\gamma} x^{1-\gamma}$, where γ measures risk aversion.
- It is not necessary to assume that the weights α, β add up to one. Important choice: $\beta = e^{-\delta}$, $\alpha = 1 - \beta$.

Epstein-Zin Utility: Certainty Equivalent

Epstein-Zin Utility: Certainty Equivalent

Epstein-Zin Utility: Deterministic Case

- If the consumption stream is deterministic, $CE(\mathcal{U}_{t+1}) = \mathcal{U}_{t+1}$.

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[(1 - \beta)C_t^{1-\phi} + \beta\mathcal{U}_{t+1}^{1-\phi} \right]^{\frac{1}{1-\phi}}.$$

- Iterating implies

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[(1 - \beta) \sum_{k=0}^T \beta^k C_{t+1}^{1-\phi} \right]^{\frac{1}{1-\phi}}.$$

- For deterministic consumption stream, maximizing \mathcal{U}_t is thus equivalent to maximize CRRA-utility

$$\sum_{t=0}^T \beta^t C_t^{1-\phi}.$$

- $\psi = \frac{1}{\phi}$ is the elasticity of intertemporal substitution.

Epstein-Zin Utility: Special Case $\gamma = \phi$

- In general, we obtain

$$U_t(C_t, C_{t+1}, \dots) = \left[(1 - \beta) C_t^{1-\phi} + \beta \mathbb{E}[U_{t+1}^{1-\gamma}]^{\frac{1-\phi}{1-\gamma}} \right]^{\frac{1}{1-\phi}}.$$

- If $\gamma = \phi$

$$U_t(C_t, C_{t+1}, \dots) = \left[(1 - \beta) C_t^{1-\phi} + \beta \mathbb{E}[U_{t+1}^{1-\phi}] \right]^{\frac{1}{1-\phi}}.$$

- Maximizing U_t is thus equivalent to maximize CRRA-utility

$$\sum_{t=0}^T \beta^t \mathbb{E}[C_t^{1-\phi}].$$

- Risk aversion γ and EIS are thus related via $\text{EIS} = \psi = 1/\gamma$.

Epstein-Zin Utility: Resolution of Uncertainty

- Consider again the following consumption streams
 - 1 In each period $t = 0, 1, \dots$, consumption is i.i.d. with

$$\mathbb{P}(C_t = \bar{C}) = \mathbb{P}(C_t = \underline{C}) = 0.5.$$

where $\bar{C} > \underline{C}$.

- 2 In each period $t = 1, 2, \dots$, $C'_t = C_0$ where

$$\mathbb{P}(C'_0 = \bar{C}) = \mathbb{P}(C'_0 = \underline{C}) = 0.5.$$

- Consider the utility of consumption stream **2**.
- There are only two possible states. In either state $i \in \{g, b\}$, the consumption stream is constant and $U_{i,t} = U_{i,t+1}$.

$$\begin{aligned} U_i^{1-\phi} &= (1-\beta)C_i^{1-\phi} + \beta(U_i^{1-\gamma})^{\frac{1-\phi}{1-\gamma}} \\ &= (1-\beta)C_i^{1-\phi} + \beta U_i^{1-\phi} \iff U_i = C_i \end{aligned}$$

Epstein-Zin Utility: Resolution of Uncertainty

- Therefore, utility of consumption stream **2** is

$$u_i^{1-\phi} = (1 - \beta)C_i^{1-\phi} + \beta\left(\frac{1}{2}\overline{C}^{1-\gamma} + \frac{1}{2}\underline{C}^{1-\gamma}\right)^{\frac{1-\phi}{1-\gamma}}$$

- Utility of consumption stream **1** is

$$u_i^{1-\phi} = (1 - \beta)C_i^{1-\phi} + \beta\left(\frac{1}{2}\overline{U}^{1-\gamma} + \frac{1}{2}\underline{U}^{1-\gamma}\right)^{\frac{1-\phi}{1-\gamma}}$$

- Consider the case $\phi > \gamma > 1$. Compare the two certainty equivalents (Jensen's inequality):

$$\left(\frac{1}{2}\overline{U}^{1-\gamma} + \frac{1}{2}\underline{U}^{1-\gamma}\right)^{\frac{1-\phi}{1-\gamma}} \geq \frac{1}{2}\overline{U}^{1-\phi} + \frac{1}{2}\underline{U}^{1-\phi}$$

Epstein-Zin Utility: Resolution of Uncertainty

- Consequently,

$$\underline{u}^{1-\phi} \geq (1-\beta)\underline{c} + \beta\left(\frac{1}{2}\bar{u}^{1-\phi} + \frac{1}{2}\underline{u}^{1-\phi}\right)$$

$$\bar{u}^{1-\phi} \geq (1-\beta)\bar{c} + \beta\left(\frac{1}{2}\bar{u}^{1-\phi} + \frac{1}{2}\underline{u}^{1-\phi}\right)$$

- Summing up and rearranging terms yield

$$\frac{1}{2}\bar{u}^{1-\phi} + \frac{1}{2}\underline{u}^{1-\phi} \geq \frac{1}{2}\bar{c}^{1-\phi} + \frac{1}{2}\underline{c}^{1-\phi}$$

- or equivalently $CE_1 \geq CE_2$.
- Therefore, if $EIS < 1/\gamma$, the agent prefers the first consumption stream and thus prefers late resolution of uncertainty.
- The opposite is true for $EIS > 1/\gamma$. For CRRA-utility ($EIS = 1/\gamma$), the agent is indifferent between early and late resolution of uncertainty.

Epstein-Zin Utility: Summary

- Time-additive utility is too restrictive to distinguish between EIS and risk aversion or to model preferences for the resolution of uncertainty.
- Certainty equivalent takes attitudes towards risk into account:
 $CE(\mathcal{U}_{t+1}) = G^{-1}(\mathbb{E}_t[G(\mathcal{U}_{t+1})])$, where $G(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ where γ is risk-aversion.
- Aggregator: CES-function with elasticity of substitution ψ .
- Utility Index:

$$\mathcal{U}_t = \left[\alpha C_t^{1-1/\psi} + \beta \left(\mathbb{E}_t \left[\mathcal{U}_{t+1}^{1-\gamma} \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}}.$$

- Typically, $\alpha = (1 - \beta)$ and $\beta = e^{-\delta}$.
- CRRA is special case if $\gamma = \frac{1}{\psi}$.
- For deterministic consumption streams, γ does not matter.

Epstein-Zin Utility: Summary

- $\theta = \frac{1-\gamma}{1-1/\psi}$ indicates preferences for resolution of uncertainty. If $\theta < 1$ ($\theta > 1$)
 - the agent has preferences for early (late) resolution of uncertainty.
 - The agent cares more (less) about uncertainty across states than about smoothing over time.
- CRRA, i.e., $\theta = 1$ implies that the agent is indifferent between early and late resolution of uncertainty.
- Risk aversion γ determines the optimal investment strategy.
 - hedging motive dominates speculation motive
 - investor takes a short position in good state variables
- EIS $\psi = 1/\phi$ determines the optimal consumption and saving behavior.
- If $\psi > 1$
 - variation over time: substitution effect dominates wealth effect
 - when investment opportunities improve, the investor saves more and consumes less

- It is a common consensus that risk aversion is greater than 1.
- Evidence on EIS is mixed:
 - Bansal and Yaron (2004) and Vissing-Joergensen and Attanasio (2003) combine equity and consumption data and estimate an EIS of 1.5 and a risk aversion in the range between 8 and 10.
 - Hall (1988), Campbell (1999), Vissing-Joergensen (2002) estimate an EIS well below one.
- Due to the lack of evidence and for reasons of tractability, many authors use unit EIS.

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Optimization Problem

- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0, \dots, T}$ modeling information.
- Agent chooses consumption and investment at $t = 0, \dots, T$ to maximize the utility index \mathcal{U} .
- Portfolio holdings $\pi^i = \frac{\varphi^i S^i}{X}$ add up to one

$$\sum_{i=0}^n \pi^i = 1.$$

- Investor's wealth $X = X^{\varphi, C}$ evolves

$$X_{t+1} = (X_t - C_t)R_{t+1}^{\pi}$$

- where the portfolio return is given by

$$R_{t+1}^{\pi} = \sum_{i=0}^n \pi_{t+1}^i R_{t+1}^i = R_{t+1}^0 + \sum_{i=1}^n \pi_{t+1}^i (R_{t+1}^i - R_{t+1}^0)$$

Optimization Problem

- The optimization problem is given by

$$J_t = \max_{c, \pi} \left[\alpha C_t^{1-1/\psi} + \beta \left(\mathbb{E}_t \left[J_{t+1}^{1-\gamma} \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}}$$

- Conjecture: The indirect utility function is given by $J_t = h_t X_t$.
- h_t captures dependence on time and state variables.
- The indirect utility function is thus

$$\begin{aligned} h_t X_t &= \left[\alpha C_t^{1-1/\psi} + \beta \left(\mathbb{E}_t \left[h_{t+1}^{1-\gamma} X_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}} \\ &= \left[\alpha C_t^{1-1/\psi} + \beta (X_t - C_t)^{1-1/\psi} \left(\mathbb{E}_t \left[(h_{t+1} R_{t+1}^\pi)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}} \\ &= C_t \left[\alpha + \beta \left(\frac{X_t - C_t}{C_t} \right)^{1-1/\psi} \left(\mathbb{E}_t \left[(h_{t+1} R_{t+1}^\pi)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}} \end{aligned}$$

First Order Condition w.r.t. Consumption

- **1.) The FOC** is given by

$$\alpha C_t^{-1/\psi} - \beta(X_t - C_t)^{-1/\psi} (\mathbb{E}_t[h_{t+1}^{1-\gamma} (R_{t+1}^\pi)^{1-\gamma}])^{1/\theta} = 0$$

- and can be expressed as

$$\alpha \left(\frac{X_t - C_t}{C_t} \right)^{1/\psi} = \beta \left(\mathbb{E}_t \left[(h_{t+1} R_{t+1}^\pi)^{1-\gamma} \right] \right)^{\frac{1}{\theta}}$$

- Remember the indirect utility function

$$h_t X_t = C_t \left[\alpha + \beta \left(\frac{X_t - C_t}{C_t} \right)^{1-1/\psi} \left(\mathbb{E}_t \left[(h_{t+1} R_{t+1}^\pi)^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}}.$$

- **2.) Substitute the FOC into J_t**

$$\begin{aligned}h_t X_t &= C_t \left[\alpha + \alpha \left(\frac{X_t - C_t}{C_t} \right)^{1-1/\psi} \left(\frac{X_t - C_t}{C_t} \right)^{1/\psi} \right]^{\frac{1}{1-1/\psi}} \\ &= C_t \left[\alpha + \alpha \left(\frac{X_t - C_t}{C_t} \right) \right]^{\frac{1}{1-1/\psi}} \\ &= C_t \left[\alpha \left(\frac{X_t}{C_t} \right) \right]^{\frac{1}{1-1/\psi}}\end{aligned}$$

- Or equivalently

$$\begin{aligned}h_t &= \frac{C_t}{X_t} \left[\alpha \left(\frac{X_t}{C_t} \right) \right]^{\frac{1}{1-1/\psi}} \\ &= \alpha^{\frac{1}{1-1/\psi}} \left(\frac{C_t}{X_t} \right)^{1-\frac{1}{1-1/\psi}}\end{aligned}$$

Indirect Utility Function

- **3.) Express J_t in terms of the consumption-wealth ratio.**

Consequently, the indirect utility function is given by

$$J_t = h_t X_t = \alpha^{\frac{1}{1-1/\psi}} \left(\frac{C_t}{X_t} \right)^{\frac{1}{1-\psi}} X_t.$$

- h_t determines how much of the current wealth is used for consumption.
- For $\psi > 1$, the indirect utility function is increasing in the wealth-consumption ratio
 - assume that investment opportunities have improved
 - $\psi > 1$ implies: consumption today decreases, consumption tomorrow increases
 - thus: wealth-consumption ratio today increases
 - consequently: higher wealth-consumption ratio signals better investment opportunities and thus higher indirect utility
- The opposite is true for $\psi < 1$.

First-Order Condition for Consumption

- **4.) Substitute h_{t+1} into the FOC for C_t and simplify.** Target: Derive something that looks like an Euler condition.

$$\begin{aligned}\alpha\left(\frac{X_t - C_t}{C_t}\right)^{1/\psi} &= \beta\left(\mathbb{E}_t\left[(h_{t+1}R_{t+1}^\pi)^{1-\gamma}\right]\right)^{\frac{1}{\theta}} \\ &= \beta\left(\mathbb{E}_t\left[\alpha^\theta\left(\frac{C_{t+1}}{X_{t+1}}\right)^{1-\gamma-\theta}(R_{t+1}^\pi)^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\end{aligned}$$

- Remember the **budget constraint**

$$X_{t+1} = (X_t - C_t)R_{t+1}^\pi$$

- Therefore,

$$\begin{aligned}\alpha\left(\frac{X_t - C_t}{C_t}\right)^{1/\psi} &= \beta\left(\mathbb{E}_t\left[\alpha^\theta\left(\frac{C_{t+1}}{(X_t - C_t)R_{t+1}^\pi}\right)^{1-\gamma-\theta}(R_{t+1}^\pi)^{1-\gamma}\right]\right)^{\frac{1}{\theta}} \\ &= \beta\left(\mathbb{E}_t\left[\alpha^\theta\left(\frac{C_{t+1}}{X_t - C_t}\right)^{1-\gamma-\theta}(R_{t+1}^\pi)^\theta\right]\right)^{\frac{1}{\theta}}\end{aligned}$$

First-Order Condition for Consumption

- Dividing by α

$$\begin{aligned}\left(\frac{X_t - C_t}{C_t}\right)^{1/\psi} &= \beta \left(\mathbb{E}_t \left[\left(\frac{C_{t+1}}{X_t - C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^\theta \right] \right)^{\frac{1}{\theta}} \\ &= \beta \left(\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} \left(\frac{C_t}{X_t - C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^\theta \right] \right)^{\frac{1}{\theta}} \\ &= \beta \left(\frac{C_t}{X_t - C_t} \right)^{-1/\psi} \left(\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^\theta \right] \right)^{\frac{1}{\theta}}\end{aligned}$$

- Therefore,

$$\begin{aligned}1 &= \beta \left(\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^\theta \right] \right)^{\frac{1}{\theta}} \\ \iff 1 &= \mathbb{E}_t \left[e^{-\delta\theta} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^\theta \right]\end{aligned}$$

First-Order Condition for Investment I

- Remember the **portfolio return**

$$R_{t+1}^{\pi} = R_{t+1}^0 + \sum_{i=1}^n \pi_{t+1}^i (R_{t+1}^i - R_{t+1}^0)$$

- 5.) The FOC w.r.t. π^i , $i = 1, \dots, n$ is given by**

$$\mathbb{E}_t \left[h_{t+1}^{1-\gamma} (R_{t+1}^{\pi})^{-\gamma} (R_{t+1}^i - R_{t+1}^0) \right] = 0,$$

- substituting the expression for h_{t+1} and the budget constraint and some algebra yields

$$\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^{\pi})^{\theta-1} (R_{t+1}^i - R_{t+1}^0) \right] = 0,$$

First-Order Condition for Investment II

- Multiplying by the portfolio weight π_t^i and summing up over $i = 0, \dots, n$

$$\mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^{\theta-1} (R_{t+1}^\pi - R_{t+1}^0) \right] = 0,$$

- Therefore,

$$\begin{aligned} \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^{\theta-1} R_{t+1}^0 \right] &= \mathbb{E}_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^\theta \right] \\ &= \beta^{-\theta} \end{aligned}$$

- where the second $=$ comes from optimal consumption. Hence, the Euler condition for asset 0 is:

$$1 = \mathbb{E}_t \left[\beta^\theta \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^{\theta-1} R_{t+1}^0 \right]$$

Pricing Kernel

- Let $\beta = e^{-\delta}$ and repeat the same steps for the other assets:

$$1 = \mathbb{E}_t \left[e^{-\delta\theta} \left(\frac{C_{t+1}}{C_t} \right)^{1-\gamma-\theta} (R_{t+1}^\pi)^{\theta-1} R_{t+1}^i \right]$$

for all assets $i = 0, \dots, n$.

- Hence we have proven:

Pricing Kernel for EZ-Preferences

The pricing kernel for EZ-Preferences is given by

$$M_{t,t+1} = e^{-\delta\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma+1-\theta} (R_{t+1}^\pi)^{\theta-1}$$

where (C, π) denotes the agent's optimal consumption and portfolio strategy.

- The log pricing kernel is thus

$$\begin{aligned}m_{t,t+1} &= \log M_{t,t+1} \\ &= -\delta\theta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1)r_{t+1}^\pi\end{aligned}$$

- where Δc_{t+1} is log consumption growth and $r_{t+1}^\pi = \Delta x_{t+1}$ is the log return on optimal wealth
- Consumption claim / Optimal wealth is an asset paying consumption as dividends

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- So far, consumption and investment have been determined endogenously.
- Now, we consider a representative agent with recursive preferences and optimal consumption growth Δc which is exogenous.
- Simplest case
 - No state variables
 - Consumption growth is i.i.d. and follows a normal distribution

$$\Delta c_{t+1} = \mu_c + \eta_{t+1}, \quad \eta_{t+1} \sim \mathcal{N}(0, \sigma_c^2)$$

- Wealth growth is i.i.d. and follows a normal distribution

$$\Delta x_{t+1} = \mu_x + \xi_{t+1}, \quad \xi_{t+1} \sim \mathcal{N}(0, \sigma_x^2)$$

Pricing the Consumption Claim

- Wealth is the price of the consumption claim. Pricing equation

$$\begin{aligned} X_t &= \mathbb{E}_t[M_{t,t+1}X_{t+1}] \\ \iff 1 &= \mathbb{E}_t[e^{m_{t,t+1} + \Delta x_{t+1}}] \\ &= \mathbb{E}_t \left[e^{-\delta\theta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta-1)\Delta x_{t+1} + \Delta x_{t+1}} \right] \\ &= \mathbb{E}_t \left[e^{-\delta\theta - \frac{\theta}{\psi} \Delta c_{t+1} + \theta \Delta x_{t+1}} \right] \\ &= e^{-\delta\theta - \frac{\theta}{\psi} \mu_c + \theta \mu_x + 0.5 \frac{\theta^2}{\psi^2} \sigma_c^2 + 0.5 \theta^2 \sigma_x^2 - \frac{\theta^2}{\psi} \sigma_{c,x}} \end{aligned}$$

- Consequently, the following condition must hold

$$\mu_x = \delta + \frac{1}{\psi} \mu_c - \frac{1}{2} \frac{\theta}{\psi^2} \sigma_c^2 - \frac{1}{2} \theta \sigma_x^2 + \frac{\theta}{\psi} \sigma_{c,x}.$$

- The risk-free asset satisfies the following pricing equation

$$\begin{aligned}1 &= \mathbb{E}_t[e^{m_{t,t+1} + r_t^f}] \\&= \mathbb{E}_t\left[e^{-\delta\theta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)\Delta x_{t+1} + r_t^f}\right] \\&= e^{-\delta\theta - \frac{\theta}{\psi}\mu_c + (\theta-1)\mu_x + 0.5\frac{\theta^2}{\psi^2}\sigma_c^2 + 0.5(\theta-1)^2\sigma_x^2 - \frac{\theta(\theta-1)}{\psi}\sigma_{c,x} + r_t^f}\end{aligned}$$

- Therefore

$$r_t^f = \delta\theta + \frac{\theta}{\psi}\mu_c - (\theta-1)\mu_x - \frac{1}{2}\frac{\theta^2}{\psi^2}\sigma_c^2 - \frac{1}{2}(\theta-1)^2\sigma_x^2 + \frac{\theta(\theta-1)}{\psi}\sigma_{c,x}$$

- Substituting μ_x implies (standard as in CRRA, new due to EZ)

$$r_t^f = \delta + \frac{1}{\psi}\mu_c - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 - \frac{1}{2}(1-\theta)\sigma_x^2.$$

Pricing of an Arbitrary Asset

- Consider an asset with return $r_{t+1}^i \sim \mathcal{N}(\mu_i, \sigma_i^2)$.
- The pricing equation is

$$\begin{aligned} 1 &= \mathbb{E}_t[e^{m_{t,t+1} + r_t^i}] \\ &= \mathbb{E}_t\left[e^{-\delta\theta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta-1)\Delta x_{t+1} + r_t^i}\right] \end{aligned}$$

- Therefore, its expected return is

$$\begin{aligned} \mu_i &= \delta\theta + \frac{\theta}{\psi}\mu_c - (\theta-1)\mu_x - \frac{1}{2}\frac{\theta^2}{\psi^2}\sigma_c^2 - \frac{1}{2}(\theta-1)^2\sigma_x^2 - 0.5\sigma_i^2 \\ &\quad + \frac{\theta(\theta-1)}{\psi}\sigma_{c,x} + \frac{\theta}{\psi}\sigma_{i,c} - (\theta-1)\sigma_{i,x}. \end{aligned}$$

- Substituting μ_x implies (standard as in CRRA, new due to EZ)

$$\text{rp}_t^i = \mu_i + 0.5\sigma_i^2 - r_t^f = \frac{\theta}{\psi}\sigma_{i,c} + (1-\theta)\sigma_{i,x}.$$

Applications of Recursive Utility

- So far we have shown how recursive utility allows to break the link between risk aversion and EIS.
- These preferences are very useful in asset pricing, portfolio choice, and are also prevalent in macroeconomics.
- They can also be used to address the other puzzles mentioned in the literature (Epstein and Zin (1989), Gilboa and Schmeidler (1989,1993), Ghirardato et al. (2004), Andries (2013))
- However
 - EZ preferences do not resolve the equity premium puzzle (Weil, 1989)
 - We need something more: long-run risk