Agenda

Issues with Time-Additive Utility

Timing of uncertainty resolution

- An agent with additive utility is indifferent between early or late resolution of uncertainty.
- Consider two consumption streams
  1. In each period $t = 0, 1, \dots, T$, consumption is i.i.d. with
     $$\mathbb{P}(C_t = \overline{C}) = \mathbb{P}(C_t = \underline{C}) = 0.5.$$
     where $\overline{C} > \underline{C}$.
  2. In each period $t = 1, 2, \dots, T$, $C'_t = C_0$ where
     $$\mathbb{P}(C'_0 = \overline{C}) = \mathbb{P}(C'_0 = \underline{C}) = 0.5.$$
- With additive utility, both streams generate the same indirect utility (check!).
- If you prefer one of them, you cannot have time-additive utility!

Issues with Time-Additive Utility

Intertemporal Substitution vs. Risk Aversion

- Agents typically dislike fluctuations in their consumption streams over time
- Suppose $C = \frac{1}{2}(\overline{C} + \underline{C})$. Consider three consumption streams
  1. Consumption is constant $C_t = C$ for all $t = 0, 1, \ldots, T$
  2. Consumption varies over time $C'_t = \overline{C}$ if $t = 0 \bmod 2$ and $C'_t = \underline{C}$ if $t = 1 \bmod 2$
  3. Consumption varies across states In each period $t = 1, 2, \ldots, T$, consumption is i.i.d.

$$\mathbb{P}(C_t = \overline{C}) = \mathbb{P}(C_t = \underline{C}) = 0.5.$$

- Agents (typically) prefer C over C' due to their aversion against intertemporal variation.
- Agents (typically) prefer C over C'' due to their aversion against variation across states (risk).

Issues with Time-Additive Utility III

Intertemporal Substitution vs. Risk Aversion

- For time additive utility, both is determined by the concavity of the utility function, e.g., CRRA-utility: $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$.
  - Relative risk aversion is given by

$$RRA = -\frac{Cu''(C)}{u'(C)} = \gamma$$

  - Elasticity of intertemporal substitution measures the responsiveness of the growth rate of consumption to the real interest rate (Hall 1988).

$$EIS = \frac{\mathrm{d}\Delta c_{t+1}}{\mathrm{d}r_t^f} = \cdots = \frac{1}{\gamma}$$

    - Substitution Effect: If r^f goes up, the agent might reduce consumption and saves more to increase future consumption.
    - Wealth Effect: If r^f goes up, the agent might feel wealthier and consumes more.
- Both properties are inseparably tied together.

Elasiticity of Intertemporal Substitution

$$EIS = \frac{d\Delta c_{t+1}}{d r_t^f}$$

$$r_t^f = \delta + \gamma \mu_c - \frac{1}{2}\gamma^2 \sigma_c^2$$
$$= \delta + \gamma(\Delta c_{t+1} - \sigma_c v_{t+1}) - \frac{1}{2}\gamma^2\sigma_c^2$$

$$\Rightarrow \Delta c_{t+1} = \frac{r_t^f - \delta + \frac{1}{2}\gamma^2\sigma_c^2}{\gamma} + \sigma_c v_{t+1}$$

$$\frac{d\Delta c_{t+1}}{d r_t^f} = \frac{1}{\gamma} = \boxed{EIS = \frac{1}{RRA}}$$

Elasiticity of Intertemporal Substitution

$$r_f^f = \delta + \frac{1}{\psi}\mu_c - \frac{1}{2}\gamma^2\sigma_c^2$$

$\underline{EIS = \psi}$

Example

① $EIS > 1$, e.g., $\psi = 2$

$\Rightarrow r_f^f$ increases by 1%

$$\Rightarrow \frac{d\Delta c_{t+1}}{d r_f^f} = 2 \Rightarrow d\Delta c_{t+1} = 2 \cdot \overbrace{d r_f^f}^{1\%}$$

$$= 2\%$$

Here, the substitution effect dominates the wealth effect.

(2) EIS < 1, $\psi = \frac{1}{2}$

r_t^f increases by 1%

Δc_{t+1} increases by just 0.5%

$\Rightarrow$ wealth effect dominates the substitution effect.

Possible Solution

$$W(X,Y) = \left(\alpha_1 X^{1-\phi} + \alpha_2 Y^{1-\phi}\right)^{\frac{1}{1-\phi}}$$

Recursive Utility

- Recursive Utility is one possible way of addressing some of the previous issues.
- A recursive utility index $\mathcal{U}$ can be expressed as

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = W_t\big(C_t, \underbrace{\mathcal{U}_{t+1}(C_{t+1}, C_{t+2}, \dots)}\big)$$

  where W is an intertemporal aggregator.
- W describes the aggregation of present consumption and future utility.
- The aggregator takes utility from current consumption C_t and expected utility from future consumption $\mathcal{U}_{t+1}$ into account.

Example: Time-Additive Utility

- Choose the linear aggregator

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = u(C_t) + \mathrm{e}^{-\delta}\mathbb{E}_t\left[\mathcal{U}_{t+1}(C_{t+1}, C_{t+2}, \dots)\right]$$

- Then, time-t utility is given by

$\mathbb{E}_{t+1}$

$$\mathcal{U}_t = u(C_t) + \mathrm{e}^{-\delta}\mathbb{E}_t\left[u(C_{t+1}) + \mathrm{e}^{-\delta}\mathbb{E}_t[\mathcal{U}_{t+2}]\right]$$

$\mathbb{E}_{t+2}$

$$\begin{aligned}
&= u(C_t) + \mathrm{e}^{-\delta}\mathbb{E}_t[u(C_{t+1})] + \mathrm{e}^{-2\delta}\mathbb{E}_t\left[u(C_{t+2}) + \mathrm{e}^{-\delta}\mathbb{E}_t[\mathcal{U}_{t+3}]\right] \\
&= u(C_t) + \mathrm{e}^{-\delta}\mathbb{E}_t[u(C_{t+1})] + \mathrm{e}^{-2\delta}\mathbb{E}_t[u(C_{t+2})] + \mathrm{e}^{-3\delta}\mathbb{E}_t[\mathcal{U}_{t+3}] \\
&= \dots \\
&= \sum_{k=0}^{T} \mathrm{e}^{-\delta k}\mathbb{E}_t[u(c_{t+k})]
\end{aligned}$$

$$\mathbb{E}_t\left[\mathbb{E}_{t+1}[X]\right] = \mathbb{E}_t[X]$$

- Standard time-additive utility is a special case of recursive utility for a linear aggregator.

Agenda

Epstein-Zin Utility

- Consider the following CES aggregator

EIS $= \frac{1}{\phi}$

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[\alpha C_t^{1-\phi} + \beta \mathrm{CE}_t(\mathcal{U}_{t+1})^{1-\phi}\right]^{\frac{1}{1-\phi}}$$

where $\phi > 0$ and

$$\mathrm{CE}_t(\mathcal{U}_{t+1}) = G^{-1}\big(\mathbb{E}_t[G(\mathcal{U}_{t+1})]\big)$$

for increasing and concave functions G.

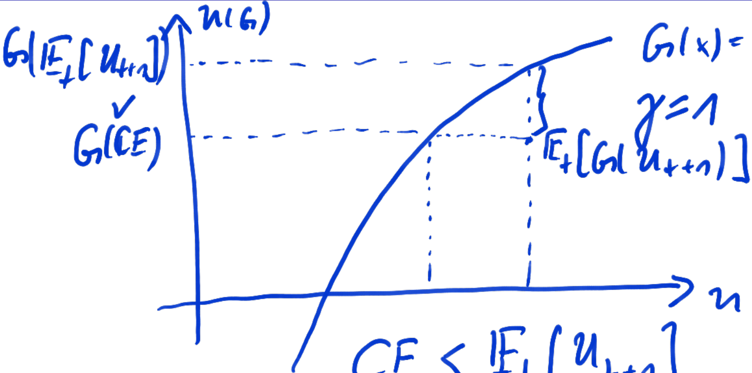
- The more concave G is, and the more uncertain the consumption stream is, the lower is the certainty equivalent.
- Most of the literature assumes $G(x) = \frac{1}{1-\gamma}x^{1-\gamma}$, where γ measures risk aversion.
- It is not necessary to assume that the weights α, β add up to one. Important choice: $\beta = \mathrm{e}^{-\delta}$, $\alpha = 1 - \beta$.

Epstein-Zin Utility: Certainty Equivalent

CE is utility generated by a consumption stream that makes the agent indifferent between: (i) taking the det. consumption stream
(ii) playing the gamble and taking the chances of higher future consumption

$$\underbrace{G(CE_t)}_{\text{utility generated by this certified det. CS}} = \underbrace{\mathbb{E}_t\left[G(U_{t+1})\right]}_{\text{expected utility of taking the deal}}$$

Epstein-Zin Utility: Certainty Equivalent



$G(x) = \log(x)$

$\gamma = 1$

$CE < \mathbb{E}_t[u_{t+1}]$

$$\underbrace{G(\mathbb{E}_t[u_{t+1}])}_{\text{risk-adjusted exp. utility}} > G(CE) = \underbrace{\mathbb{E}_t[G(u_{t+1})]}_{\text{exp. risk-adjusted utility}}$$

Epstein-Zin Utility: Deterministic Case

- If the consumption stream is deterministic, $\mathrm{CE}(\mathcal{U}_{t+1}) = \mathcal{U}_{t+1}$.

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[(1-\beta)C_t^{1-\phi} + \beta\mathcal{U}_{t+1}^{1-\phi}\right]^{\frac{1}{1-\phi}}.$$

- Iterating implies

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[(1-\beta)\boxed{\sum_{k=0}^{T}\beta^k C_{t+1}^{1-\phi}}\right]^{\frac{1}{1-\phi}}.$$

- For deterministic consumption stream, maximizing $\mathcal{U}_t$ is thus equivalent to maximize CRRA-utility

$$\sum_{t=0}^{T}\beta^t C_t^{1-\phi}.$$

- $\psi = \frac{1}{\phi}$ is the elasticity of intertemporal substitution.

Epstein-Zin Utility: Special Case $\gamma = \phi$ $= \frac{1}{\psi} = \frac{1}{EIS}$

- In general, we obtain

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[(1-\beta)C_t^{1-\phi} + \beta\mathbb{E}[\mathcal{U}_{t+1}^{1-\gamma}]^{\frac{1-\phi}{1-\gamma}}\right]^{\frac{1}{1-\phi}}.$$

- If $\gamma = \phi$

$$\mathcal{U}_t(C_t, C_{t+1}, \dots) = \left[(1-\beta)C_t^{1-\phi} + \beta\mathbb{E}[\mathcal{U}_{t+1}^{1-\phi}]\right]^{\frac{1}{1-\phi}}.$$

- Maximizing $\mathcal{U}_t$ is thus equivalent to maximize CRRA-utility

$$\sum_{t=0}^{T} \beta^t \mathbb{E}[C_t^{1-\phi}].$$

- Risk aversion γ and EIS are thus related via EIS $= \psi = 1/\gamma$.

- Consider again the following consumption streams
  1. In each period $t = 0, 1, \ldots,$ consumption is i.i.d. with
  $$\mathbb{P}(C_t = \overline{C}) = \mathbb{P}(C_t = \underline{C}) = 0.5.$$
  where $\overline{C} > \underline{C}$.
  2. In each period $t = 1, 2, \ldots,$ $C'_t = C_0$ where
  $$\mathbb{P}(C'_0 = \overline{C}) = \mathbb{P}(C'_0 = \underline{C}) = 0.5.$$
- Consider the utility of consumption stream **2**.
- There are only two possible states. In either state $i \in \{g, b\}$, the consumption stream is constant and $\mathcal{U}_{i,t} = \mathcal{U}_{i,t+1}$.

$$\mathcal{U}_i = \left((1-\beta)C_i^{1-\phi} + \beta\, \mathbb{E}[\mathcal{U}_{t+1}^{1-\gamma}]^{\frac{1-\phi}{1-\gamma}}\right)^{\frac{1}{1-\phi}}$$

$$\mathcal{U}_i^{1-\phi} = (1-\beta)C_i^{1-\phi} + \beta(\mathcal{U}_i^{1-\gamma})^{\frac{1-\phi}{1-\gamma}}$$

$$\mathcal{U}_i^{1-\phi} = (1-\beta)C_i^{1-\phi} + \beta\mathcal{U}_i^{1-\phi} \iff \boxed{\mathcal{U}_i = C_i}$$

Epstein-Zin Utility: Resolution of Uncertainty

- Therefore, utility of consumption stream **2** is

$$\mathcal{U}_i^{1-\phi} = (1-\beta)C_i^{1-\phi} + \beta\Big(\overbrace{\frac{1}{2}\overline{C}^{1-\gamma} + \frac{1}{2}\underline{C}^{1-\gamma}}^{\mathbb{E}_0[C_i^{1-\gamma}]}\Big)^{\frac{1-\phi}{1-\gamma}}$$

- Utility of consumption stream **1** is

$$\mathcal{U}_i^{1-\phi} = (1-\beta)C_i^{1-\phi} + \beta\Big(\frac{1}{2}\overline{\mathcal{U}}^{1-\gamma} + \frac{1}{2}\underline{\mathcal{U}}^{1-\gamma}\Big)^{\frac{1-\phi}{1-\gamma}}$$

- Consider the case $\phi > \gamma > 1$. Compare the two certainty equivalents (Jensen's inequality):

$\frac{1}{\psi} > \gamma > 1$

$$\Big(\frac{1}{2}\overline{\mathcal{U}}^{1-\gamma} + \frac{1}{2}\underline{\mathcal{U}}^{1-\gamma}\Big)^{\frac{1-\phi}{1-\gamma}} \geq \underbrace{\frac{1}{2}\overline{\mathcal{U}}^{1-\phi} + \frac{1}{2}\underline{\mathcal{U}}^{1-\phi}}$$

Epstein-Zin Utility: Resolution of Uncertainty

- Consequently,

$$\underline{\mathcal{U}}^{1-\phi} \geq (1-\beta)\underline{C} + \beta\Big(\frac{1}{2}\overline{\mathcal{U}}^{1-\phi} + \frac{1}{2}\underline{\mathcal{U}}^{1-\phi}\Big)$$
$$\overline{\mathcal{U}}^{1-\phi} \geq (1-\beta)\overline{C} + \beta\Big(\frac{1}{2}\overline{\mathcal{U}}^{1-\phi} + \frac{1}{2}\underline{\mathcal{U}}^{1-\phi}\Big)$$

- Summing up and rearranging terms yield

$$\boxed{\frac{1}{2}\overline{\mathcal{U}}^{1-\phi} + \frac{1}{2}\underline{\mathcal{U}}^{1-\phi}} \geq \boxed{\frac{1}{2}\overline{C}^{1-\phi} + \frac{1}{2}\underline{C}^{1-\phi}}$$

- or equivalently $\mathrm{CE}_1 \geq \mathrm{CE}_2$.
- Therefore, if $\boxed{\text{EIS} < 1/\gamma}$, the agent prefers the first consumption stream and thus prefers late resolution of uncertainty.
- The opposite is true for $\boxed{\text{EIS} > 1/\gamma}$. For CRRA-utility ($\text{EIS} = 1/\gamma$), the agent is indifferent between early and late resolution of uncertainty.

Epstein-Zin Utility: Summary

- Time-additive utility is too restrictive to distinguish between EIS and risk aversion or to model preferences for the resolution of uncertrainty.
- Certainty equivalent takes attitudes towards risk into account: $\mathrm{CE}(\mathcal{U}_{t+1}) = G^{-1}\big(\mathbb{E}_t[G(\mathcal{U}_{t+1})]\big)$, where $G(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ where γ is risk-aversion.
- Aggregator: CES-function with elasticity of substitution ψ.
- Utility Index:

$$\mathcal{U}_t = \left[\alpha C_t^{1-1/\psi} + \beta\Big(\mathbb{E}_t\Big[\mathcal{U}_{t+1}^{1-\gamma}\Big]\Big)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}.$$

- Typically, $\alpha = (1-\beta)$ and $\beta = \mathrm{e}^{-\delta}$. , Alternatively $\alpha = 1$, $\beta = e^{-\delta}$
- CRRA is special case if $\gamma = \frac{1}{\psi}$.
- For deterministic consumption streams, γ does not matter.

Epstein-Zin Utility: Summary

- $\theta = \frac{1-\gamma}{1-1/\psi}$ indicates preferences for resolution of uncertainty. If $\theta < 1$ ($\theta > 1$)
  - the agent has preferences for early (late) resolution of uncertainty.
  - The agent cares more (less) about uncertainty across states than about smoothing over time.
- CRRA, i.e., $\theta = 1$ implies that the agent is indifferent between early and late resolution of uncertainty.
- Risk aversion γ determines the optimal investment strategy.

  $\gamma > 1$
  - hedging motive dominates speculation motive
  - investor takes a short position in good state variables
- EIS $\psi = 1/\phi$ determines the optimal consumption and saving behavior.
- If $\psi > 1$
  - variation over time: substitution effect dominates wealth effect
  - when investment opportunities improve, the investor saves more and consumes less

Evidence on RRA and EIS

- It is a common consensus that risk aversion is greater than 1.
- Evidence on EIS is mixed:
  - Bansal and Yaron (2004) and Vissing-Joergensen and Attanasio (2003) combine equity and consumption data and estimate an EIS of 1.5 and a risk aversion in the range between 8 and 10.
  - Hall (1988), Campbell (1999), Vissing-Joergenen (2002) estimate an EIS well below one.
- Due to the lack of evidence and for reasons of tractability, many authors use unit EIS.

Agenda

Optimization Problem

- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0,\dots,T}$ modeling information.
- Agent chooses consumption and investment at $t = 0, \dots, T$ to maximize the utility index $\mathcal{U}$.
- Portfolio holdings $\pi^i = \frac{\varphi^i S^i}{X}$ add up to one

fraction of wealth invested in asset i

$$\sum_{i=0}^{n} \pi^i = 1.$$

$$R^{\pi}_{t+1} = r^{\pi}_{t+1} + 1$$

- Investor's wealth $X = X^{\varphi, C}$ evolves

$$X_{t+1} = (X_t - C_t) R^{\pi}_{t+1}$$

- where the portfolio return is given by

$$R^{\pi}_{t+1} = \sum_{i=0}^{n} \pi^i_{t+1} R^i_{t+1} = R^0_{t+1} + \sum_{i=1}^{n} \pi^i_{t+1} (R^i_{t+1} - R^0_{t+1})$$

Optimization Problem

- The optimization problem is given by

$J = \max_{C,\pi} U$

$$J_t = \max_{c,\pi} \left[\alpha C_t^{1-1/\psi} + \beta \left(\mathbb{E}_t \left[J_{t+1}^{1-\gamma} \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right]^{\frac{1}{1-1/\psi}}$$

- Conjecture: The indirect utility function is given by $J_t = h_t X_t$.
- h_t captures dependence on time and state variables.
- The indirect utility function is thus

$X_{t+1} = (X_t - C_t) R^{\pi}_{t+1}$

$$h_t X_t = \left[\alpha C_t^{1-1/\psi} + \beta \left(\mathbb{E}_t \left[h_{t+1}^{1-\gamma} X_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}}$$

$$= \frac{C_t}{C_t} \left[\alpha C_t^{1-1/\psi} + \beta (X_t - C_t)^{1-1/\psi} \left(\mathbb{E}_t \left[(h_{t+1} R^{\pi}_{t+1})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}}$$

$$= C_t \left[\alpha + \beta \underbrace{\left(\frac{X_t - C_t}{C_t} \right)}^{1-1/\psi} \left(\mathbb{E}_t \left[(h_{t+1} R^{\pi}_{t+1})^{1-\gamma} \right] \right)^{\frac{1}{\theta}} \right]^{\frac{1}{1-1/\psi}}$$

First Order Condition w.r.t. Consumption

- **1.) The FOC** is given by

$$\alpha C_t^{-1/\psi} - \beta(X_t - C_t)^{-1/\psi}\left(\mathbb{E}_t[h_{t+1}^{1-\gamma}(R_{t+1}^{\pi})^{1-\gamma}]\right)^{1/\theta} = 0$$

- and can be expressed as

$$\alpha\left(\frac{X_t - C_t}{C_t}\right)^{1/\psi} = \beta\left(\mathbb{E}_t\left[(h_{t+1}R_{t+1}^{\pi})^{1-\gamma}\right]\right)^{\frac{1}{\theta}}$$

- Remember the indirect utility function

$$h_t X_t = C_t\left[\alpha + \beta\left(\frac{X_t - C_t}{C_t}\right)^{1-1/\psi}\left(\mathbb{E}_t\left[(h_{t+1}R_{t+1}^{\pi})^{1-\gamma}\right]\right)^{\frac{1}{\theta}}\right]^{\frac{1}{1-1/\psi}}.$$

Indirect Utility Function

- **2.) Substitute the FOC into J_t**

$$h_t X_t = C_t \left[\alpha + \alpha \left(\frac{X_t - C_t}{C_t} \right)^{1-1/\psi} \left(\frac{X_t - C_t}{C_t} \right)^{1/\psi} \right]^{\frac{1}{1-1/\psi}}$$

$$= C_t \left[\alpha + \alpha \left(\frac{X_t - C_t}{C_t} \right) \right]^{\frac{1}{1-1/\psi}}$$

$$= C_t \left[\alpha \left(\frac{X_t}{C_t} \right) \right]^{\frac{1}{1-1/\psi}} \quad \Big| : X_t$$

- Or equivalently

$$J_t = h_t \cdot X_t$$

$$h_t = \frac{C_t}{X_t} \left[\alpha \left(\frac{X_t}{C_t} \right) \right]^{\frac{1}{1-1/\psi}}$$

$$= \alpha^{\frac{1}{1-1/\psi}} \left(\frac{C_t}{X_t} \right)^{1 - \frac{1}{1-1/\psi}}$$

Indirect Utility Function

- **3.) Express J_t in terms of the consumption-wealth ratio.**
  Consequently, the indirect utility function is given by

$$J_t = h_t X_t = \boxed{\alpha^{\frac{1}{1-1/\psi}} \left(\frac{C_t}{X_t}\right)^{\frac{1}{1-\psi}} X_t}.$$

- h_t determines how much of the current wealth is used for consumption.
- For $\psi > 1$, the indirect utility function is increasing in the wealth-consumption ratio
  - assume that investment opportunities have improved
  - $\psi > 1$ implies: consumption today decreases, consumption tomorrow increases
  - thus: wealth-consumption ratio today increases
  - consequently: higher wealth-consumption ratio signals better investment opportunities and thus higher indirect utility
- The opposite is true for $\psi < 1$.

First-Order Condition for Consumption

- **4.) Substitute h_{t+1} into the FOC for C_t and simplify**. Target: Derive something that looks like an Euler condition.

$$\alpha\Big(\frac{X_t - C_t}{C_t}\Big)^{1/\psi} = \beta\Big(\mathbb{E}_t\Big[(h_{t+1}R^{\pi}_{t+1})^{1-\gamma}\Big]\Big)^{\frac{1}{\theta}}$$

$$= \beta\Big(\mathbb{E}_t\Big[\alpha^{\theta}\Big(\frac{C_{t+1}}{X_{t+1}}\Big)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{1-\gamma}\Big]\Big)^{\frac{1}{\theta}}$$

- Remember the **budget constraint**

$$X_{t+1} = (X_t - C_t)R^{\pi}_{t+1}$$

$$1 = \mathbb{E}_t\Big[e^{-\delta}\frac{u_c(C_{t+1})}{u_c(C_t)}R_{t+1}\Big]$$

- Therefore,

$$\alpha\Big(\frac{X_t - C_t}{C_t}\Big)^{1/\psi} = \beta\Big(\mathbb{E}_t\Big[\alpha^{\theta}\Big(\frac{C_{t+1}}{(X_t - C_t)R^{\pi}_{t+1}}\Big)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{1-\gamma}\Big]\Big)^{\frac{1}{\theta}}$$

$$= \beta\Big(\mathbb{E}_t\Big[\alpha^{\theta}\Big(\frac{C_{t+1}}{X_t - C_t}\Big)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta}\Big]\Big)^{\frac{1}{\theta}}$$

First-Order Condition for Consumption

$$\theta = \frac{1-\gamma}{1-1/\psi}$$

- Dividing by α

$$\begin{aligned}
\underbrace{\left(\frac{X_t - C_t}{C_t}\right)^{1/\psi}} &= \beta\left(\mathbb{E}_t\left[\left(\frac{C_{t+1}}{X_t - C_t}\right)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta}\right]\right)^{\frac{1}{\theta}} \\
&= \beta\left(\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma-\theta}\underbrace{\left(\frac{C_t}{X_t - C_t}\right)}^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta}\right]\right)^{\frac{1}{\theta}} \\
&= \beta\underbrace{\left(\frac{C_t}{X_t - C_t}\right)}^{-1/\psi}\left(\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta}\right]\right)^{\frac{1}{\theta}}
\end{aligned}$$

- Therefore,

$$1 = \mathbb{E}_t\left[\underbrace{e^{-\delta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma\theta}\left(R^{\pi}_{t+1}\right)^{\theta-1}}\, R^{\pi}_{t+1}\right]$$

$$1 = \beta\left(\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta}\right]\right)^{\frac{1}{\theta}} \quad M_{t,t+1}$$

$$\iff \quad 1 = \mathbb{E}_t\left[e^{-\delta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta}\right] \quad \Big| \cdot e^{-\delta\theta}$$

$$e^{\delta\theta} = \mathbb{E}_t\left[\left(C_{t+1}/C_t\right)^{1-\gamma-\theta}\left(R^{\pi}\right)^{\theta}\right]$$

First-Order Condition for Investment I

- Remember the **portfolio return**

$$R^{\pi}_{t+1} = R^0_{t+1} + \sum_{i=1}^{n} \pi^i_{t+1}(R^i_{t+1} - R^0_{t+1})$$

- **5.) The FOC w.r.t. π^i, $i = 1, \dots, n$ is given by**

$$\mathbb{E}_t\left[h^{1-\gamma}_{t+1}(R^{\pi}_{t+1})^{-\gamma}(R^i_{t+1} - R^0_{t+1})\right] = 0,$$

- substituting the expression for h_{t+1} and the budget constraint and some algebra yields

$$\mathbb{E}_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta-1}\underbrace{(R^i_{t+1} - R^0_{t+1})}\right] = 0,$$

First-Order Condition for Investment II

- Multiplying by the portfolio weight π_t^i and summing up over $i = 0, \dots, n$

$$\mathbb{E}_t\Big[\Big(\frac{C_{t+1}}{C_t}\Big)^{1-\gamma-\theta}(R_{t+1}^\pi)^{\theta-1}(R_{t+1}^\pi - R_{t+1}^0)\Big] = 0,$$

- Therefore,

$$\begin{aligned}\mathbb{E}_t\Big[\Big(\frac{C_{t+1}}{C_t}\Big)^{1-\gamma-\theta}(R_{t+1}^\pi)^{\theta-1}R_{t+1}^0\Big] &= \mathbb{E}_t\Big[\Big(\frac{C_{t+1}}{C_t}\Big)^{1-\gamma-\theta}(R_{t+1}^\pi)^\theta\Big] \\ &= \beta^{-\theta} \quad 1 \cdot \beta^{+\theta}\end{aligned}$$

- where the second = comes from optimal consumption. Hence, the Euler condition for asset 0 is:

$\beta = e^{-\delta}$

$$\boxed{1 = \mathbb{E}_t\Big[\beta^\theta\Big(\frac{C_{t+1}}{C_t}\Big)^{1-\gamma-\theta}(R_{t+1}^\pi)^{\theta-1}R_{t+1}^0\Big]}$$

Pricing Kernel

- Let $\beta = \mathrm{e}^{-\delta}$ and repeat the same steps for the other assets:

$$1 = \mathbb{E}_t\left[\underbrace{\mathrm{e}^{-\delta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma-\theta}(R^{\pi}_{t+1})^{\theta-1}} R^{i}_{t+1}\right]$$

  for all assets $i = 0, \ldots, n$.

- Hence we have proven:

Pricing Kernel for EZ-Preferences

The pricing kernel for EZ-Preferences is given by

$$M_{t,t+1} = \mathrm{e}^{-\delta\theta}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma+1-\theta}\boxed{(R^{\pi}_{t+1})^{\theta-1}}$$

where (C, π) denotes the agent's optimal consumption and portfolio strategy.

in case of CRRA

$\theta = 1$ $\Rightarrow M_{t,t+1} = e^{-\delta}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$

Log Pricing Kernel

$$-\gamma + 1 - \theta = (1-\gamma) - \frac{1-\gamma}{1-1/\psi} = (1-\gamma)\left[1 - \frac{1}{1-1/\psi}\right]$$

$$= -\frac{1}{\psi}\,\frac{1-\gamma}{1-1/\psi} = -\frac{\theta}{\psi}$$

- The log pricing kernel is thus

$$\begin{aligned} m_{t,t+1} &= \log M_{t,t+1} \\ &= -\delta\theta - \frac{\theta}{\psi}\Delta c_{t+1} + (\theta - 1) r^{\pi}_{t+1} \end{aligned}$$

- where Δc_{t+1} is log consumption growth and $r^{\pi}_{t+1} = \Delta x_{t+1}$ is the log return on optimal wealth
- Consumption claim / Optimal wealth is an asset paying consumption as dividends

Agenda

Model Setup

- So far, consumption and investment have been determined endogenously.
- Now, we consider a representative agent with recursive preferences and optimal consumption growth Δc which is exogenous.
- Simplest case
  - No state variables
  - Consumption growth is i.i.d. and follows a normal distribution

$$\Delta c_{t+1} = \mu_c + \eta_{t+1}, \qquad \eta_{t+1} \sim \mathcal{N}(0, \sigma_c^2)$$

  - Wealth growth is i.i.d. and follows a normal distribution

$$\Delta x_{t+1} = \mu_x + \xi_{t+1}, \qquad \xi_{t+1} \sim \mathcal{N}(0, \sigma_x^2)$$

Pricing the Consumption Claim

$X_t = \mathbb{E}_t[M_{t,t+1}(X_{t+1} + C_t)]$

- Wealth is the price of the consumption claim. Pricing equation

$$X_t = \mathbb{E}_t[M_{t,t+1}X_{t+1}] \qquad 1 : X_t$$

$$\iff e^0 = 1 = \mathbb{E}_t[\mathrm{e}^{m_{t,t+1}+\Delta x_{t+1}}]$$

$$= \mathbb{E}_t\left[\mathrm{e}^{-\delta\theta-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)\Delta x_{t+1}+\Delta x_{t+1}}\right]$$

$$= \mathbb{E}_t\left[\mathrm{e}^{-\delta\theta-\frac{\theta}{\psi}\Delta c_{t+1}+\theta\Delta x_{t+1}}\right]$$

$$= \mathrm{e}^{-\delta\theta-\frac{\theta}{\psi}\mu_c+\theta\mu_x+0.5\frac{\theta^2}{\psi^2}\sigma_c^2+0.5\theta^2\sigma_x^2-\frac{\theta^2}{\psi}\sigma_{c,x}}$$

- Consequently, the following condition must hold

$$\mu_x = \delta + \frac{1}{\psi}\mu_c - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 - \frac{1}{2}\theta\sigma_x^2 + \frac{\theta}{\psi}\sigma_{c,x}.$$

Risk-Free Rate

- The risk-free asset satisfies the following pricing equation

$\theta = \frac{1-\gamma}{1-1/\psi}$

$$
\begin{aligned}
1 &= \mathbb{E}_t[\mathrm{e}^{m_{t,t+1}+r_t^f}] \\
&= \mathbb{E}_t\left[\mathrm{e}^{-\delta\theta-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)\Delta x_{t+1}+r_t^f}\right] \\
&= \mathrm{e}^{-\delta\theta-\frac{\theta}{\psi}\mu_c+(\theta-1)\mu_x+0.5\frac{\theta^2}{\psi^2}\sigma_c^2+0.5(\theta-1)^2\sigma_x^2-\frac{\theta(\theta-1)}{\psi}\sigma_{c,x}+r_t^f}
\end{aligned}
$$

- Therefore

$$
r_t^f = \delta\theta + \frac{\theta}{\psi}\mu_c - \boxed{(\theta-1)\mu_x} - \frac{1}{2}\frac{\theta^2}{\psi^2}\sigma_c^2 - \frac{1}{2}(\theta-1)^2\sigma_x^2 + \frac{\theta(\theta-1)}{\psi}\sigma_{c,x}
$$

- Substituting μ_x implies (standard as in CRRA, new due to EZ)

$$
r_t^f = \delta + \frac{1}{\psi}\mu_c - \frac{1}{2}\frac{\theta}{\psi^2}\sigma_c^2 - \boxed{\frac{1}{2}\underbrace{(1-\theta)}_{+}\sigma_x^2}.
$$

Pricing of an Arbitrary Asset

- Consider an asset with return $r^i_{t+1} \sim \mathcal{N}(\mu_i, \sigma_i^2)$.
- The pricing equation is

$$\begin{aligned} 1 &= \mathbb{E}_t[\mathrm{e}^{m_{t,t+1}+r^i_t}] \\ &= \mathbb{E}_t\left[\mathrm{e}^{-\delta\theta-\frac{\theta}{\psi}\Delta c_{t+1}+(\theta-1)\Delta x_{t+1}+r^i_t}\right] \end{aligned}$$

- Therefore, its expected return is

$$\begin{aligned} \mu_i = \delta\theta + \frac{\theta}{\psi}\mu_c - (\theta-1)\mu_x - \frac{1}{2}\frac{\theta^2}{\psi^2}\sigma_c^2 - \frac{1}{2}(\theta-1)^2\sigma_x^2 - 0.5\sigma_i^2 \\ + \frac{\theta(\theta-1)}{\psi}\sigma_{c,x} + \frac{\theta}{\psi}\sigma_{i,c} - (\theta-1)\sigma_{i,x}. \end{aligned}$$

- Substituting μ_x implies (standard as in CRRA, new due to EZ)

$$\mathrm{rp}^i_t = \mu_i + 0.5\sigma_i^2 - r^f_t = \frac{\theta}{\psi}\sigma_{i,c} + (1-\theta)\sigma_{i,x}.$$

Applications of Recursive Utility

- So far we have shown how recursive utility allows to break the link between risk aversion and EIS.
- These preferences are very useful in asset pricing, portfolio choice, and are also prevalent in macroeconomics.
- They can also be used to address the other puzzles mentioned in the literature (Epstein and Zin (1989), Gilboa and Schmeidler (1989,1993), Ghirardato et al. (2004), Andries (2013))
- However
  - EZ preferences do not resolve the equity premium puzzle (Weil, 1989)
  - We need something more: long-run risk