Agenda

1 Option Pricing in Partial Equilibrium

2 General Equilibrium Asset Pricing
- Introduction
- Time-Additive Utility
- Consumption-based CAPM (CCAPM)
- Example: Log-Normal Case
- Asset Pricing Puzzles

3 Habit Formation and Asset Pricing

Why General Equilibrium?

- So far, we have taken asset returns as given and derived prices of derivative claims.
- A general equilibrium model seeks to explain the asset pricing dynamics.
- Empirical findings (e.g., Bansal and Yaron 2004)
  - expected return on government bonds $\mathbb{E}[r_f] = 0.86\%$
  - standard deviation of return on government bonds: $\sigma(r_f) = 0.97\%$
  - equity risk premium: $\mathbb{E}[r_m - r_f] = 6.33\%$
  - standard deviation of equity return: $\sigma(r_m) = 19.42\%$
- These and other facts must be explained and not just taken as given.

General Equilibrium

- An **equilibrium** consists of
  - consumption and investment decision of each investor
  - prices of all traded assets

  such that
  - each investor maximizes his utility
  - markets clear (demand = supply)
- In equilibrium, asset returns materialize **endogenously** from supply and demand.

General Equilibrium: First Outline

1. Assume that prices are given
2. Solve portfolio planning problem for each investor result: optimal portfolio holdings and optimal consumption (partial equilibrium)
3. Check market clearing condition
   - aggregate demand for assets = aggregate supply
   - aggregate consumption = aggregate endowment
4. Choose asset prices such that markets clear.

- Representative Investor must hold the whole stock market
- no investment in the riskfree asset

Representative Investor

- Aggregate demand depends on
    - individual demand of each investor
    - which depends on initial endowment, preferences, beliefs, ...
- Aggregation might be rather involved (we will deal with heterogeneous investors and consumption sharing rules later)
- Easy solution and standard approach: representative investor
    - one investor who represents the market
    - equilibrium condition: representative investor has to consume aggregate endowment

Racall: CAPM

- Static model with 1 period, Returns are normally distributed $\mathcal{N}(\mu_i, \sigma_i^2)$
- μ-σ-optimization
  - ↳ optimal asset allocation (see H. Markowitz)
- all market participants hold the market portfolio
- derive the SML

$$\mu_i = r_f + \beta_i(\mu_m - r_f)$$

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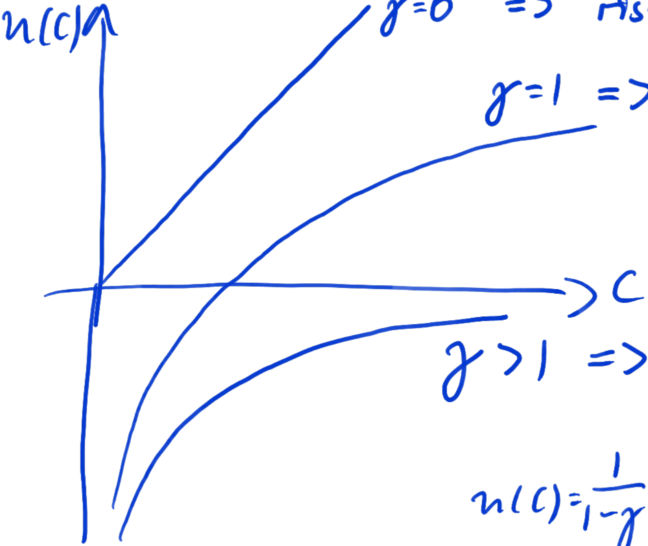
Representative Agent

- Planning horizon T.
- Representative agent
  - decides on consumption $C = (C_t)_{t=0,\dots,T}$ and investment $\varphi = (\varphi_t)_{t=0,\dots,T}$ at time $t = 0, \dots, T$.
  - gains additive utility from the consumption stream; utility index

$$\mathcal{U} = \sum_{t=0}^{T} \mathrm{e}^{-\delta t} \mathbb{E}\left[u(C_t)\right]$$

  - δ is the pure rate of time preferences.
  - u is a utility function with $u'(C) > 0$ (agents are greedy) and $u''(C) < 0$ (utility saturates).
  - Popular Choice: Power Utility (CRRA) $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$

Popular Choice: CRRA-utility



$\gamma = 0 \Rightarrow$ risk-neutral case

$\gamma = 1 \Rightarrow$ log-utility

$$u(c) = \log(c)$$

$\gamma > 1 \Rightarrow$ risk averse

$$u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$$

Popular Choice: CRRA-utility

- Popular utility function

$$u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$$

- The degree of relative risk aversion (Arrow 1970; Pratt 1966) is given by

$$\begin{aligned} \mathrm{RRA} &= -C \frac{u''(C)}{u'(C)} \\ &= -C \frac{-\gamma C^{-\gamma-1}}{C^{-\gamma}} \\ &= \gamma. \end{aligned}$$

- For $\gamma = 1$, the preferences collapse to log-utility $u(C) = \log(C)$ which often leads to closed-form solutions.
- However, empirical evidence suggests $\gamma > 1$.

Proof – Log-utility

$$\lim_{\gamma \to 1} \frac{C^{1-\gamma} - 1}{1-\gamma} \qquad \text{L'Hospital's rule:}$$

$$\Rightarrow \lim_{\gamma \to 1} \frac{-C^{1-\gamma} \cdot \log(C)}{-1} = \log(C).$$

More General: HARA-utility

largest class of "tractable" utility functions

$$u(C) = \frac{1-\gamma}{\gamma}\left(\frac{aC}{1-\gamma} + b\right)^{\gamma}$$

- $\underline{\gamma = 1}$: risk-neutral case
- $\underline{\gamma = 2}$: quadratic utility (CAPM)
- $\underline{b = 1, \gamma \to -\infty}$: $u(c) = \frac{1}{a}(1 - e^{-aC})$
- CRRA: $\gamma < 1, a = 1-\gamma, b = 0$
- log-utility additionally $\gamma \to 0$.

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Consumption-based CAPM (CCAPM) – Model Setup

- To understand the mechanics, we consider a one-period model with one asset with price S_t and return r_{t+1}
- The agents optimization problem is then

$$\max_{C,\varphi} \left\{ u(C_t) + \mathrm{e}^{-\delta}\mathbb{E}_t\left[u(C_{t+1})\right] \right\}$$

subject to

$$\begin{aligned} C_t &= E_t - \varphi S_t \\ C_{t+1} &= E_{t+1} + \varphi S_t(1 + \cancel{r_t}) \end{aligned}$$

$$r_{t+1}$$

Consumption-based CAPM (CCAPM) – Euler Condition I

- To solve the decision problem, we consider its Lagrangian

$$
\begin{aligned}
\mathcal{L}(C_t, C_{t+1}, \varphi, \lambda_t, \lambda_{t+1}) = {} & u(C_t) + \mathbb{E}_t[\mathrm{e}^{-\delta} u(C_{t+1})] \\
& - \lambda_t[C_t - E_t + \varphi S_t] - \lambda_{t+1}[C_{t+1} - E_{t+1} - \varphi S_t(1 + r_{t+1})]
\end{aligned}
$$

- Taking first-order conditions

$$
\begin{aligned}
\mathcal{L}_{C_t} &= u_c(C_t) - \lambda_t = 0 \\
\mathcal{L}_{C_{t+1}} &= \mathbb{E}_t[\mathrm{e}^{-\delta} u_c(C_{t+1})] - \lambda_{t+1} = 0 \\
\mathcal{L}_{\varphi} &= -\lambda_t + \lambda_{t+1}(1 + r_{t+1}) = 0
\end{aligned}
$$

$$\lambda_{t+1} = \frac{\lambda_t}{1+r_t}$$

- Therefore,

$$\lambda_t = u_c(C_t), \qquad \lambda_{t+1} = \frac{u_c(C_t)}{1 + r_{t+1}}$$

Consumption-based CAPM (CCAPM) – Euler Condition II

- This implies

$$\mathbb{E}\left[\frac{M_{t+1}}{M_t}\cdot S_{t+1}\right] = S_t$$

$$\mathbb{E}_t[\mathrm{e}^{-\delta}u_c(C_{t+1})] = \frac{u_c(C_t)}{1+r_t}. \quad \Leftrightarrow \mathbb{E}[M_{t+1}S_{t+1}] = M_t S_t$$

- Consequently, we end up with the pricing equation

Euler Condition

PK at t+1 / PK at t

$$\mathbb{E}_t\left[\overbrace{\mathrm{e}^{-\delta}\frac{u_c(C_{t+1})}{u_c(C_t)}}(1+r_{t+1})\right] = 1.$$

- This can also be achieved in multi-period settings and with more than one asset (exercise!)
- Therefore, the pricing kernel is given by

$$M_{t,t+1} = \mathrm{e}^{-\delta}\frac{u_c(C_{t+1})}{u_c(C_t)}.$$

where $M_{t,t+1} = M_{t+1}/M_t$.

Consumption-based CAPM (CCAPM) – Asset Pricing

- Risk-free rate

$$\overbrace{e^{-r_t^f}}^{\text{cont. compounding}} = \overbrace{\frac{1}{1+r_t^f}}^{\text{discrete compounding}} = \mathbb{E}_t\left[\mathrm{e}^{-\delta}\frac{u_c(C_{t+1})}{u_c(C_t)}\right].$$

- Risk premium

$$\mathrm{rp}_t = \mathbb{E}_t[r_{t+1}] - r_t^f = -(1+r_t^f)\overbrace{\mathrm{cov}_t[M_{t,t+1}, r_{t+1}]}^{<0}.$$

- Typically, marginal utility is negatively correlated with asset returns.
  - Investors demand a positive risk premium to hold this asset.
  - Asset is positively correlated with consumption.
  - The opposite is true if asset returns are positively correlated with MU.

Proof – Risk Premium

$$\mathrm{cov}(X,Y) = \mathbb{E}[X\cdot Y] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$$

$$\mathrm{cov}_t(M_{t,t+1}, R_{t+1}) = \overbrace{\mathbb{E}_t[M_{t,t+1}\cdot R_{t+1}]}^{=1} - \boxed{\underbrace{\mathbb{E}_t[M_{t,t+1}]}_{=1/1+r_t^f}}\,\mathbb{E}_t[R_{t+1}]$$

$$R_{t+1} = 1 + r_{t+1}$$

$$\Rightarrow \mathrm{cov}_t(M_{t,t+1}, R_{t+1}) = 1 - \frac{1}{1+r_t^f}\,\mathbb{E}_t[1+r_{t+1}]$$

$$\Rightarrow (1+r_t^f)\,\mathrm{cov}_t(M_{t,t+1}, R_{t+1}) = \underbrace{1+r_t^f - \mathbb{E}_t(1+r_{t+1})}$$

$$\Rightarrow rp_{t+1} = -(1+r_t^f)\cdot\mathrm{cov}_t(M_{t,t+1}, R_{t+1})$$

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Notational Convention

- We use capital letters for a process or variable, e.g., C_t stands for consumption, D_t for dividends,...
- We use small letters for the logarithm of this variable, e.g., $c_t = \ln C_t$.
- We use the operator Δ for the time increment, e.g., $\Delta C_{t+1} = C_{t+1} - C_t$
- Notice that for any process X, we have

$$X_{t+1} = X_t \mathrm{e}^{\Delta x_{t+1}}.$$

Example: Log-Normal Case

Lucas-Tree Model

- Investor with CRRA utility $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$, $\gamma \neq 1$.
- Assumption: Log consumption growth is normally distributed

$$\Delta c_{t+1} = \mu_c + \nu_{t+1}$$

where μ_c denotes expected consumption growth and $\nu_{t+1} \sim_{i.i.d.} \mathcal{N}(0, \sigma_c^2)$.

- From the Euler condition, we obtain the pricing kernel

$$M_{t,t+1} = \mathrm{e}^{-\delta} \frac{u_c(C_{t+1})}{u_c(C_t)} = \mathrm{e}^{-\delta - \gamma \Delta c_{t+1}}$$

Example: Log-Normal Case

- Risk-free rate

$$e^{-r_t^f} = \mathbb{E}_t[M_{t,t+1}] = \mathbb{E}_t\left[e^{-\delta-\gamma\Delta c_{t+1}}\right]$$

Therefore,

$$\boxed{r_t^f = \delta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2}$$

  - δ represents the role of discounting.
  - $\gamma\mu_c$ represents intertemporal consumption smoothing.
  - $-\frac{1}{2}\gamma^2\sigma_c^2$ represents precautionary savings.
- Risky asset with i.i.d. returns $r_{t+1} \sim \mathcal{N}(\mu_i, \sigma_i^2)$

$$\mathbb{E}_t\left[e^{-\delta-\gamma\Delta c_{t+1}+r_{t+1}}\right] = 1$$

and thus

$$\mathrm{rp}_t^i = \mu_i + \frac{1}{2}\sigma_i^2 - r_t^f = \gamma\,\mathrm{cov}(\Delta c_{t+1}, r_{t+1}) = \gamma\sigma_{c,i}.$$

Proof: Risk-free Rate

$$\mathbb{E}_t[M_{t,t+1}] = \boxed{e^{-r_t^f}} = \mathbb{E}_t\left[e^{-\delta - \gamma \Delta c_{t+1}}\right]$$

$$\boxed{\begin{array}{l} \mathbb{E}[e^{\mu + \sigma z}] \\ [z \sim \mathcal{N}(0,1)] \\ = e^{\mu + \frac{1}{2}\sigma^2} \\ \neq e^{\mu} = med(e^{\mu+\sigma z}) \end{array}}$$

$$= \mathbb{E}_t\left[e^{-\delta - \gamma(\mu_c + v_{t+1})}\right] \quad \mathcal{N}(0, \sigma_c^2)$$

$$= \mathbb{E}_t\left[e^{-\delta - \gamma\mu_c - \gamma\sigma_c v_{t+1}}\right]$$

$$= \boxed{e^{-\delta - \gamma\mu_c + \frac{1}{2}\gamma^2\sigma_c^2}}$$

$$r_t^f = -\log \mathbb{E}_t[M_{t,t+1}]$$

$$r_t^f = \delta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2$$

Proof: Equity Premium

$$r^i_{t+1} \sim \mathcal{N}(\mu_i, \sigma_i^2) \quad \text{i.i.d}$$

$$\mathbb{E}_t\left[M_{t,t+1} \cdot e^{r^i_{t+1}}\right] = 1 = \mathbb{E}_t\left[e^{\overbrace{-\delta - \gamma \Delta c_{t+1} + r^i_{t+1}}^{Z_{t+1}}}\right]$$

$$Z_{t+1} \sim \mathcal{N}\left(-\delta - \gamma\mu_c + \mu_i \, ; \, \gamma^2\sigma_c^2 + \sigma_i^2 - 2\gamma\sigma_{c,i}\right)$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X,Y)$$

$$\Rightarrow e^{-\delta - \gamma\mu_c + \mu_i + \frac{1}{2}\left(\gamma^2\sigma_c^2 + \sigma_i^2 - 2\gamma\sigma_{ci}\right)} = 1$$

$$-\delta - \gamma\mu_c + \mu_i + \frac{1}{2}\left(\gamma^2\sigma_c^2 + \sigma_i^2 - 2\gamma\sigma_{ci}\right) = 0$$

$$\underbrace{\mu_i - r_f}_{\text{discrete RP}} + \underbrace{\frac{1}{2}\sigma_i^2}_{\text{Jensen correction term}} = \gamma\sigma_{ci}$$

Price-Dividend Ratio

- Consumption claim X is an asset paying consumption as dividends.
- It equals financial wealth of the representative investor.
- For its price, we get

$$S_t = \mathbb{E}_t[M_{t,t+1} S_{t+1}]$$

$$X_t = \mathbb{E}_t\left[M_{t,t+1}(X_{t+1} + \underline{C_{t+1}})\right] \checkmark$$

- Solve for the price-dividend ratio $Z_t = X_t / C_t$ which is constant in this model:

$$X_t = ZC_t = \mathbb{E}_t\left[M_{t,t+1}(ZC_{t+1} + C_{t+1})\right] \quad | : C_t$$

$$Z = \mathbb{E}_t\left[M_{t,t+1}(Z+1)\frac{C_{t+1}}{C_t}\right] \quad | : (Z+1)$$

$$\frac{Z}{1+Z} = \mathbb{E}_t\left[M_{t,t+1}\mathrm{e}^{\Delta c_{t+1}}\right] \qquad M_{t,t+1} = e^{-\delta - \gamma \Delta c_{t+1}}$$

$$\frac{Z}{1+Z} = \mathbb{E}_t\left[\mathrm{e}^{-\delta-(\gamma-1)\Delta c_{t+1}}\right]$$

$$\frac{Z}{1+Z} = \mathrm{e}^{\overbrace{-\delta-(\gamma-1)\mu_c+\frac{1}{2}(\gamma-1)^2\sigma_c^2}^{(\cdots)}}$$

Price-Dividend Ratio

$$z = \frac{e^{(\dots)}}{1 - e^{(\dots)}}$$

$$y_d := \frac{1}{z} = \frac{1 - e^{(\dots)}}{e^{(\dots)}} = e^{-(\dots)} - 1$$

$$\Rightarrow \quad 1 + y_d = e^{-(\dots)} = e^{y_c}$$

$$y_c = \ln(1 + y_d)$$

$$y_c = \delta + (\gamma - 1)\mu_c - \frac{1}{2}(\gamma - 1)^2 \sigma_c^2$$

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Calibration

- The aim of asset pricing is to find a consumption-based explanation for asset pricing moments.
- Idea: Take $\mu_c \approx 0.02$, $\mu_i \approx 0.07$, $\sigma_c \approx 0.02$, $\sigma_i \approx 0.16$, $\rho_{c,i} \approx 0.2$ and calibrate the preferences.
- Risk-premium:

$rp = \gamma \cdot \sigma_{ci}$

$$\gamma = \frac{\mathrm{rp}_t^i}{\sigma_{c,i}} = \frac{\mu_i + \frac{1}{2}\sigma_i^2 - r_t^f}{\sigma_c \sigma_i \rho_{c,i}}$$
$$= \frac{0.06 + 0.5 \cdot 0.16^2}{0.02 \cdot 0.16 \cdot 0.2} > 1,000$$

- Experimental-based estimates of risk aversion in the range from 2 to 10.
- Moral: To match the risk-premium one needs an unrealistically high degree of risk-aversion

Even worse!

- Let's try to match the risk-free rate:

$$r_t^f = \delta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2 \approx \delta + \gamma\mu_c$$

- Then,

$$\gamma \approx \frac{r_t^f - \delta}{\mu_c}$$

- Typical values for δ in the range from 0.01 to 0.05. For $\delta = 0.02$, one obtains

$$\gamma = \frac{0.01 - 0.02}{0.02} = -0.5.$$

  For $\delta = 0$, one gets $\gamma = 0.5$.

- Moral: To match the risk-free rate one needs an unrealistically low degree of risk-aversion

Asset Pricing Puzzles

- Equity Premium Puzzle / Risk-free Rate Puzzle (Mehra and Prescott 1985; Weil 1993): Simple consumption-based asset pricing model cannot simultaneously explain
  1. high equity premium
  2. low risk-free rates
- Excess Volatility Puzzle (Shiller 1981): Simple consumption-based asset pricing model cannot explain high volatility of stock prices compared to the volatility of consumption/dividend growth.

Way out?

- The model considered so far is too restrictive.
  - Time-additive utility
  - Normally-distributed returns
  - Linear dynamics
- Better specification of preferences
  1. ✗ Habit Formation (Abel 1990, Campbell and Cochrane 1999)
  2. Recursive Utility (Epstein and Zin 1989)
- Better consumption / dividend dynamics
  1. ✗ Long-run risk model (Bansal and Yaron 2004)
  2. ✗ Disaster models (Barro 2006, 2009; Barro and Jin 2015; Gabaix 2008)
- Heterogeneous Agents
- Market Frictions
- Production-based asset pricing
- Partial information
- ...