Advanced Financial Economics I

– Part 1: Discrete Time Models –

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Agenda

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Primary Securities

- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- One period: $t = 0, T$.
- Asset prices are **exogenously** given by a $(n+1)$-dimensional positive process.
- Money market account S^0 with interest rate r

$$S_0^0 = 1, \qquad S_T^0 = S_0^0(1+r)$$

- n primary securities (stocks)

$$S = (S_t)_{t=0,T}, \qquad S_t = (S_t^0, \dots, S_t^n)^\top$$

where $S_0 \in \mathbb{R}^{n+1}$ and S_T is a $(n+1)$-dimensional random vector with finitely many possible outcomes.

Contingent Claims

A (contingent) claim guarantees a payoff C_T at T.

- A **Forward Contract** obligates the holder to buy (or sell) an asset at a fixed price K at a predetermined date T. Payoff profile:

$$C_T = S_T - K$$

- A **Call Option** offers the right (but not the obligation) to *buy* an asset at a fixed price K on or up to a specified date T. Payoff profile:

$$C_T = \max\{0, S_T - K\}$$

- A **Put Option** offers the right (but not the obligation) to *sell* an asset at a fixed price K on or up to a specified date T. Payoff profile:

$$C_T = \max\{0, K - S_T\}$$

Contingent Claims

Trading Strategy

- A trading strategy $\varphi = (\varphi^0, \dots, \varphi^n)^\top \in \mathbb{R}^{n+1}$ is a $(n+1)$-dimensional vector.
- φ^0: number of bonds.
- $\varphi^0 S_t^0$: money amount invested in the bond.
- φ^i: number of shares of stock $i = 1, \dots, n$.
- $\varphi^i S_t^i$: money amount invested in stock $i = 1, \dots, n$.

Financial Wealth

Financial Wealth $X_t = X_t^\varphi$ at time $t = 0, T$ is given by the portfolio value of the trading strategy

$$X_t = \varphi^\top S_t = \sum_{i=0}^{n} \varphi^i S_t^i.$$

It describes the portfolio value for an investor using the trading strategy φ.

Pricing of Contingent Claims

Definition

- A trading strategy φ is an **arbitrage opportunity** if

$$X_0^\varphi = 0, \qquad X_T^\varphi \geq 0, \qquad \mathbb{P}(X_T^\varphi > 0) > 0.$$

- A model is **arbitrage-free** if no arbitrage opportunities exist.
- A claim C is **attainable** if a trading strategy φ exists such that $X_T^\varphi = C_T$. Such a strategy is called a **replication strategy** or **hedging strategy**.
- A financial market is **complete** if and only if all contingent claims are attainable.

- We only consider arbitrage-free, but not necessarily complete models.
- In models with arbitrage opportunities, very strange things can happen.

Pricing by Replication

Theorem: Law of One Price

Suppose the market is arbitrage-free.

1. For an attainable claim C with hedging strategy φ,

   $$C_0 = X_0^{\varphi}$$

   is the unique arbitrage-free price, i.e., trading in the primary assets *and* the claim does not lead to arbitrage opportunities.

2. If $X_T^{\varphi} = X_T^{\psi}$ for trading strategies φ and ψ, then

   $$X_0^{\varphi} = X_0^{\psi}.$$

Proof: LOP

Proof: LOP

Example: Binomial Model – Model Setup

We consider the simplest possible model with only two assets

- Money Market Account $B_1 = B_0(1+r)$
- Stock $S_1 = S_0(1+y)$, where

$$p = \mathbb{P}(y = u) = 1 - \mathbb{P}(y = d) \in (0,1)$$

- $u > r > d$.
- Contingent claim with payoff:

$$C_T = c_u \mathbf{1}_{\{y=u\}} + c_d \mathbf{1}_{\{y=d\}}$$

- Example: Call Option with strike price $S_0(1+u) > K > S_0(1+d)$:

$$c_u = S_0(1+u) - K, \qquad c_d = 0$$

Example: Binomial Model – Replication

- A replication strategy has to satisfy the following linear system

$$\begin{aligned} c_u &= \varphi^0 S_0^0(1+r) + \varphi^1 S_0^1(1+u) \\ c_d &= \varphi^0 S_0^0(1+r) + \varphi^1 S_0^1(1+d) \end{aligned}$$

- This system has a unique solution

$$\varphi^0 S_0^0 = \frac{c_d(1+u) - c_u(1+d)}{(1+r)(u-d)}, \qquad \varphi^1 S_0^1 = \frac{c_u - c_d}{u-d}.$$

- The price of the claim is thus

$$C_0 = X_0^{\varphi} = \varphi^0 S_0^0 + \varphi^1 S_0^1 = \frac{q}{1+r} c_u + \frac{1-q}{1+r} c_d$$

where $q = \frac{r-d}{u-d}$.

Example: Binomial Model – Risk-Neutral Probability

- The price has thus the following representation

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{1+r}\right]$$

where $\mathbb{Q}$ is a probability measure with

$$\mathbb{Q}(Y=u) = q = 1 - \mathbb{Q}(Y=d).$$

- Such a measure is called a **risk-neutral measure**.
- Under $\mathbb{Q}$, prices can be calculated as expected discounted cashflows.
- There is no risk-premium involved.
- Asset prices satisfy the pricing relation

$$S_0^i = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T^i}{1+r}\right].$$

Change of Measure – Equivalent Probability Measures

In the binomial example, we constructed a probability measure $\mathbb{Q}$ that relates in a certain sense to $\mathbb{P}$.

Definition

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be equivalent if both measures possess the same null sets, i.e., for all events $A \in \mathcal{A}$

$$\mathbb{P}(A) = 0 \quad \Longleftrightarrow \quad \mathbb{Q}(A) = 0.$$

We use the notation $\mathbb{P} \sim \mathbb{Q}$ for equivalent probability measures.

Change of Measure – Equivalent Probability Measures

The following theorem states how to switch between two equivalent probability measures.

Theorem: Radon-Nikodym

Let $\mathbb{P} \sim \mathbb{Q}$ denote two equivalent probability measures, then there exists a unique (a.s.), positive random variable $Z = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{Q}}[Y] = \mathbb{E}^{\mathbb{P}}[ZY]$$
$$\mathbb{E}^{\mathbb{P}}[Y] = \mathbb{E}^{\mathbb{Q}}\left[\frac{Y}{Z}\right]$$

for all random variables $Y : \Omega \to \mathbb{R}$.

In our example, the process Z is just a binomial random variable given by

$$Z^u = \frac{q}{p}, \qquad Z^d = \frac{1-q}{1-p}.$$

Risk-Neutral Pricing

Definition

A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called a **risk-neutral measure** or an **equivalent martingale measure** if and only if

$$S_0^i = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_T^i}{1+r}\right]$$

for all assets $i = 0, \ldots, n$.

- Under a risk-neutral measure, one can calculate asset prices as a discounted expected value.
- The pricing problem thus collapses to the calculation of an expectation.

Risk-Neutral Pricing

Theorem: Risk-Neutral Pricing

Suppose the market is arbitrage-free. Let $\mathbb{Q}$ be a risk-neutral probability measure and let C be an attainable claim with hedging strategy φ. Then its unique arbitrage-free price is given by

$$C_0 = X_0^{\varphi} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{1+r}\right]$$

Proof: Risk-Neutral Pricing

Relation between $\mathbb{Q}$ and $\mathbb{P}$

- In general, $\mathbb{Q}$ is more pessimistic than $\mathbb{P}$.
- In reality, most investors are risk-averse and demand for a positive risk premium.
- The pricing relation is thus

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{C_T}{1+r}\right] = \mathbb{E}^{\mathbb{P}}\left[\frac{C_T}{1+r+\mathrm{rp}_T}\right]$$

- Notice that $\frac{1}{1+r+\mathrm{rp}_T} < \frac{1}{1+r}$.
- If both expectations are equal, the risk-neutral measure puts more weight on bad events.
- In the binomial example $Z^u = \frac{q}{p} < 1$, $Z^d = \frac{1-q}{1-p} > 1$.

Example: Binomial Model – Stochastic Discount Factor

- **Recall:** The price of the claim is

$$C_0 = \frac{q}{1+r} c_u + \frac{1-q}{1+r} c_d$$
$$= p \frac{q}{p(1+r)} c_u + (1-p) \frac{1-q}{(1-p)(1+r)} c_d.$$

- Therefore, the price can be expressed as

$$C_0 = \mathbb{E}^{\mathbb{P}}\Big[C_T \cdot M_T\Big]$$

where $M_T^u = \frac{q}{p(1+r)}$ and $M_T^d = \frac{1-q}{(1-p)(1+r)}$, i.e., $M_T = \frac{Z}{1+r}$.

- Such a random variable M_T is called a **stochastic discount factor** (SDF) or **pricing kernel**.
- Using the SDF, one can calculate asset prices under $\mathbb{P}$.

Pricing under $\mathbb{P}$

Definition

A random variable M_T is called a **stochastic discount factor** or a **pricing kernel** if and only if

$$S_0^i = \mathbb{E}^{\mathbb{P}}\left[S_T^i \cdot M_T\right]$$

for all assets $i = 0, \ldots, n$.

- Again, the pricing problem collapses to the calculation of an expectation.
- But: given a SDF, you can price under $\mathbb{P}$ instead of $\mathbb{Q}$.
- In the following, we use the notation $\mathbb{E}[\cdot] = \mathbb{E}^{\mathbb{P}}[\cdot]$.

Stochastic Discount Factor and the Risk-free Rate

- From the definition of the SDF, it follows that

$$S_0^0 = \mathbb{E}\Big[S_T^0 \cdot M_T\Big]$$
$$S_0^0 = \mathbb{E}\Big[S_0^0 \cdot M_T\Big](1+r)$$

- Therefore,

$$\frac{1}{1+r} = \mathbb{E}[M_T]$$

or

$$r = \frac{1}{\mathbb{E}[M_T]} - 1.$$

- This relation holds in very complicated models.

Pricing under $\mathbb{P}$

Theorem: Stochastic Discount Factor

Suppose the market is arbitrage-free. Let M be a stochastic discount factor and let C be an attainable claim with hedging strategy φ. Then its unique arbitrage-free price is given by

$$C_0 = X_0^{\varphi} = \mathbb{E}\Big[C_T \cdot M_T\Big].$$

Given a risk-neutral measure $\mathbb{Q}$, the SDF can be expressed as

$$M_T = \frac{1}{1+r}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}},$$

i.e., it reflects both discounting with the risk-free interest rate and a change of measure from $\mathbb{P}$ to $\mathbb{Q}$.

Proof: Stochastic Discount Factor

Example: Put-Call Parity

Example: Put-Call Parity

Agenda

Primary Securities in Discrete Time

- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0,\dots,T}$ modeling information.
- Trading dates: $t = 0, \dots, T$
- Asset prices are **exogenously** given by a $(n+1)$-dimensional positive adapted process.
- Money market account S^0 with interest rate r_t

$$S_0^0 > 0, \qquad S_t^0 = S_{t-1}^0(1 + r_t)$$

- n Primary securities (stocks)

$$S = (S_t)_{t=0,\dots,T}, \qquad S_t = (S_t^0, \dots, S_t^n)^\top$$

where S is a $(n+1)$-dimensional adapted process.

Sigma-Algebras, Information, and Conditional Expectations

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Trading in Discrete Time

- A trading strategy

$$\varphi = (\varphi_t)_{t=0,\dots,T}, \qquad \varphi = (\varphi_t^0, \dots, \varphi_t^n)^\top$$

  is a $(n+1)$-dimensional adapted process.
- φ_t^0: number of bonds held in $(t, t+1]$.
- φ_t^i: number of shares of stock $i = 1, \dots, n$ held in $(t, t+1]$.

Financial Wealth

Financial Wealth $X_t = X_t^\varphi$ at time t is given by

$$X_t = \varphi_t^\top S_t = \sum_{i=0}^{n} \varphi_t^i S_t^i.$$

Trading in Discrete Time

We consider an investor with the following properties:

- No exogenous income or expenses.
- Trading does not cause transaction costs or taxes.
- Changes in wealth only caused by changes in asset prices.

Definition

A trading strategy φ is **self-financing** if

$$X_t = \varphi_t^\top S_t = \varphi_{t-1}^\top S_t.$$

- Wealth after trading equals wealth before trading.

Trading in Discrete Time

For a self-financing strategy, financial wealth satisfies

$$\begin{aligned} X_{t+1}^{\varphi} &= \varphi_{t+1}^{\top} S_{t+1} \\ &= \varphi_t^{\top} S_{t+1} \\ &= \varphi_t^{\top} (S_{t+1} - S_t + S_t) \\ &= \varphi_t^{\top} S_t + \varphi_t^{\top} (S_{t+1} - S_t) \\ &= X_t^{\varphi} + \varphi_t^{\top} \Delta S_t \end{aligned}$$

Consequently,

$$\Delta X_t^{\varphi} = \varphi_t^{\top} \Delta S_t$$

and

$$X_t^{\varphi} = X_0 + \sum_{\ell=1}^{t} \Delta X_\ell^{\varphi} = X_0 + \sum_{\ell=1}^{t} \varphi_\ell^{\top} \Delta S_\ell$$

No Arbitrage and Market Completeness Revisited

Definition

- A trading strategy φ is an **arbitrage opportunity** if

$$X_0^\varphi = 0, \qquad X_T^\varphi \geq 0, \qquad \mathbb{P}(X_T^\varphi > 0) > 0.$$

- A model is **arbitrage-free** if no arbitrage opportunities exist.
- A claim C is **attainable** if there exists a *self-financing* trading strategy φ such that $X_T^\varphi = C_T$. Such a strategy is called a **replication strategy** or **hedging strategy**.
- A financial market is **complete** if and only if all contingent claims are attainable.

Pricing of Contingent Claims

- A multi-period model consists of a sequence of single-period models.
- A multi-period model is arbitrage-free if all single-period models are arbitrage-free (Delbean and Schachermeyer, 2006).
- All relevant pricing relations carry over.
- We denote the discounted asset prices by $\widetilde{S}^i$, i.e.,

$$\widetilde{S}^i_t = \frac{S^i_t}{S^0_t}.$$

Definition

A stochastic process $X = (X_t)_{t=0,\dots,T}$ is called a $\mathbb{P}$-martingale if

$$\mathbb{E}^{\mathbb{P}}[X_{t+1} \mid \mathcal{F}_t] = X_t.$$

Examples for Martingales

Examples for Martingales

Pricing of Contingent Claims

Definition

- A probability measure $\mathbb{Q} \sim \mathbb{P}$ is a **risk-neutral measure** or an equivalent martingale measure (EMM) if $\widetilde{S}^i$ is a martingale under $\mathbb{Q}$ for all $i = 0, \ldots, n$, i.e.,

$$\widetilde{S}_t^i = \mathbb{E}_t^{\mathbb{Q}}[\widetilde{S}_{t+1}^i].$$

- A non-negative stochastic process $M = (M_t)_{t=0,\ldots,T}$ is called a **stochastic discount factor** or a pricing kernel, if MS^i is a martingale under $\mathbb{P}$ for all $i = 0, \ldots, n$, i.e.,

$$S_t^i = \mathbb{E}_t[S_{t+1}^i M_{t+1}] \frac{1}{M_t}.$$

Pricing of Contingent Claims

Theorem: Pricing in Discrete Time

Assume that the market is free of arbitrage. Let C be an attainable claim with hedging strategy φ.

- Suppose that $\mathbb{Q}$ is a risk-neutral measure. Then, the unique arbitrage-free price of C is

$$C_t = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{C_T}{S_T^0}\right] S_t^0.$$

- Suppose that M is a stochastic discount factor. Then, the unique arbitrage-free price of C is

$$C_t = \mathbb{E}_t[C_T M_T]\frac{1}{M_t}.$$

Fundamental Theorems of Asset Pricing

First Fundamental Theorem of Asset Pricing

The following are equivalent

1. The market is free of arbitrage.
2. There exists a risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$.

Second Fundamental Theorem of Asset Pricing

Suppose the market is free of arbitrage. The following are equivalent

1. The market is complete.
2. There exists a unique risk-neutral probability measure $\mathbb{Q} \sim \mathbb{P}$.

Proof: FTAP 1

Proof: FTAP 1

Fundamental Theorems of Asset Pricing

- Existence and uniqueness of risk-neutral probability measure are equivalent to existence and uniqueness of stochastic discount factor.
- In continuous time, the easy directions of the FTAPs still hold, i.e.,
  1. Existence of risk-neutral probability measure $\Rightarrow$ No arbitrage.
  2. Uniqueness of risk-neutral probability measure $\Rightarrow$ Market completeness.
- For the converse directions, one has to replace the concepts of no arbitrage and EMM by somewhat more involved concepts (NFLVR, ELMM, EσMM), see Delbean and Schachermeyer (1994, 1998).

Example: Cox, Ross, Rubinstein (1979)

- The CRR model extends the binomial model.
- Bond price:

$$B_t = (1 + r)^t$$

- Stock price:

$$S_t = S_0 \prod_{t=1}^{T} (1 + y_t)$$

where y_t are iid with

$$p = \mathbb{P}(y_1 = u) = 1 - \mathbb{P}(y_1 = d)$$

- It is used in practice to approximate continuous-time models.
- One can show that the solution converges to the Black-Scholes formula as the number of time steps increases.

Example: Cox, Ross, Rubinstein (1979)

- We have shown that the one-period model is free of arbitrage if $u > r > d$.
- Therefore, the CRR model is free of arbitrage if $u > r > d$ and thus a risk-neutral measure exists.
- For a given claim one can find a replicating strategy φ by solving a system of two linear equations at each node of the event tree.
- Under the NA condition $u > r > d$ these systems have unique solutions.
- The solutions provide the replicating strategy in the corresponding state at the corresponding time.
- Consequently, the model is complete and the risk-neutral measure is unique.

Arbitrage Opportunity in CRR

Completeness