Valuation and Risk Management

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Course Information

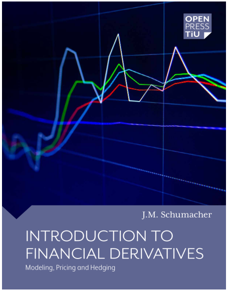
- Lecturers:
  - Christoph Hambel (financial modeling and derivative pricing)
  - Nikolaus Schweizer (numerical methods and risk measures)
- This course ...
  - ... provides an introduction to financial modeling, pricing, and risk management beyond the Black-Scholes framework
  - ... requires some knowledge from mathematics and finance, especially from stochastic calculus (Wiener process, Itô's Lemma, Change of measure, Girsanov's Theorem, ...)
  - ... contains a guest lecture by (tba)
- Grading:
  - Exam 70%
  - Two Assignments (15% each)

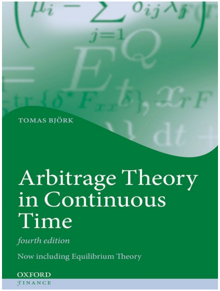
What to expect?

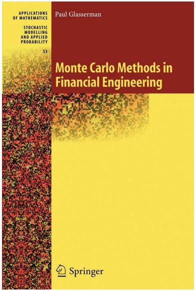
- What can you expect from us? We will...
  - ... timely provide the learning material on Canvas
  - ... also upload the slides with hand-written complements (some slides are intentionally blank)
  - ... illustrate the lecture by examples
  - ... provide problem sets and a sample exam to practice the material
  - ... be available for questions
  - ... offer a virtual Q&A session after the last lecture
- What will we expect from you? You should ...
  - ... be well-prepared when you come to the lecture
  - ... actively participate in the lecture
  - ... take the opportunity and ask us questions during the classes

Recommended Literature

- We do not make any book the *mandatory* reading for this course. However, we highly recommend the following textbooks:
  - Schumacher, J.M.: *Introduction to Financial Derivatives: Modeling, Pricing and Hedging* (Open Press TiU)
  - Björk, T.: *Arbitrage Theory in Continuous Time (Oxford)*
  - Glasserman, P.: *Monte-Carlo Methods in Financial Engineering* (Springer)
- This course follows the notation in Schumacher (2020), which contains a lot of exercises.







Preliminary Schedule

Please notice that the plan can change!

- Mon, 28.08.2023, 12:45, WZ105
- Mon, 04.09.2023, 12:45, WZ105
- Tue, 05.09.2023, 14:45, CUBE 218
- Mon, 11.09.2023, 12:45, WZ105
- Mon, 18.09.2023, 12:45, WZ105
- Tue, 19.09.2023, 14:45, CUBE 218
- Mon, 25.09.2023, 12:45, WZ105
- Mon, 02.10.2023, 12:45, WZ105
- Tue, 03.10.2023, 14:45, CUBE 218
- Mon, 09.10.2023, 12:45, WZ105
- Tue, 10.10.2023, 14:45, CUBE 218

Structure of the Course (First Half)

1. Introduction to Financial Modeling
   - Discrete vs. Continuous Time Modeling
   - Fundamentals from Stochastic Calculus
2. Continuous time: Generic State Space Model
   - Framework
   - No Arbitrage and the First FTAP
   - The Numéraire-dependent Pricing Formula
   - Replication and the Second FTAP
   - The PDE Approach
3. Contingent Claim Pricing
   - Black-Scholes Revisited
   - Option Pricing in Incomplete Markets
   - Models with Dividends

Structure of the Course (First Half)

4. Fixed Income Modeling
   - Bonds and Yields
   - Interest Rates and Interest Rate Derivatives
   - Short Rate Models for the TSIR
   - Empirical Models
   - The Heath-Jarrow-Morton Framework
   - LIBOR Market Model and Option Pricing
5. A Brief Introduction to Credit Risk
   - Reduced-Form Modeling
   - Merton's Firm Value Model

Part I

Introduction to Financial Modeling

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Time

- Discrete time with time horizon T:

$$t \in \{0, \Delta t, 2\Delta t, \dots, (n-1)\Delta t, \underbrace{n\Delta t}_{=T}\} = \{i\Delta t \mid i = 0, \dots, n\}$$

- Continuous time as a limit of discrete time ($\Delta t \to 0$ as $n \to \infty$):

$$t \in [0, T]$$

Modeling in Discrete Time: First Idea

- Risk-free asset (bond) paying a constant interest rate:

$$B_{t+\Delta t} = B_t(1 + r \cdot \Delta t) \qquad \Longleftrightarrow \qquad \frac{\Delta B_{t+\Delta t}}{B_t} = r \cdot \Delta t$$

- Risky asset (stock):

$$S_{t+\Delta t} = S_t(1 + \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}), \qquad \nu_{t+\Delta t} \sim_{i.i.d.} (0, 1)$$

- Return:

$$\frac{\Delta S_{t+\Delta t}}{S_t} = \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

Problem: Returns are not necessarily bounded from below by -1 and thus asset prices can be negative.

Log Returns

- Way out? $\rightarrow$ Model log returns, L_t, and take the exponential:

$$S_{t+\Delta t} = S_t \mathrm{e}^{\Delta L_{t+\Delta t}}$$

- Risk-free asset (bond):

$$B_{t+\Delta t} = B_t \mathrm{e}^{r \cdot \Delta t} \quad \Longleftrightarrow \quad r\Delta t = \ln\left(\frac{B_{t+\Delta t}}{B_t}\right) = \Delta \ln B_{t+\Delta t}$$

- Risky asset (stock):

$$\Delta L_{t+\Delta t} = \ln(S_{t+\Delta t}) - \ln(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

- Now, we take the limit to continuous time, i.e., we increase the number of periods ($n \rightarrow \infty$) while keeping the time horizon constant, i.e., $\Delta t = \frac{T}{n} \rightarrow 0$.

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$$
\begin{aligned}
S_T &= S_0 \prod_{i=0}^{n-1} \mathrm{e}^{\Delta L_{(i+1)\Delta t}} \\
&= S_0 \exp\left\{ \sum_{i=0}^{n-1} \left[\left(\mu - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma \cdot \nu_{(i+1)\Delta t} \cdot \sqrt{\Delta t} \right] \right\} \\
&= S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{\Delta t} \cdot \sum_{i=1}^{n} \nu_{i\Delta t} \right\} \\
&= S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{T} \cdot \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i\Delta t} \right\}
\end{aligned}
$$

According to the CLT: $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \nu_{i\Delta t} \rightarrow_d Z_T \sim \mathcal{N}(0,1)$ as $n \rightarrow \infty$, i.e.,

$$
S_T \rightarrow_d S_0 \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) T + \sigma \cdot \sqrt{T} \cdot Z_T \right\}
$$

From Discrete to Continuous Time

- In the limit, the log return is normally distributed:

$$L_T = L_0 + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma \cdot \sqrt{T} \cdot Z_T$$

- Consequently, in the limit S_T is log-normally distributed with

$$\text{mean:} \quad \mathbb{E}[S_T] = S_0 e^{\mu \cdot T}$$

$$\text{variance:} \quad \text{var}(S_T) = S_0^2 e^{2\mu \cdot T}[e^{\sigma^2 T} - 1]$$

- Does this mean that any discrete-time model converges to a log-normal distribution?

- How can we model asset prices in continuous time?

Trading in Discrete Time

- Assume that there is a frictionless financial market (i.e., no taxes, no transaction costs, no short-selling constraints, ...)
- Throughout the lecture we will be using vector notation:

$$m : \text{number of basic assets}$$

$$Y_t : m\text{-dimensional vector of asset prices at time } t$$

$$\phi_t : \text{vector of number of units of assets held at time } t$$

- Portfolio value generated by the *portfolio strategy* (or *trading strategy*) ϕ:

$$V_t = \phi_t' Y_t.$$

- A portfolio strategy ϕ is *self-financing* if trading neither generates nor destroys money, i.e.,

$$\phi_{t-\Delta t}' Y_t = \phi_t' Y_t.$$

- Suppose that rebalancing takes place at times $0 < t_1 < \cdots < t_n = T$, i.e., $t_j = j\Delta t$.

$$
\begin{aligned}
V_T &= V_0 + \sum_{j=0}^{n-1} (V_{t_{j+1}} - V_{t_j}) && \text{(telescope rule)} \\
&= V_0 + \sum_{j=0}^{n-1} \phi'_{t_j} (Y_{t_{j+1}} - Y_{t_j}) && \text{(self-financing portfolio)} \\
&= V_0 + \sum_{j=0}^{n-1} \phi'_{t_j} \, \Delta Y_{t_{j+1}}.
\end{aligned}
$$

- The sum $\sum_{j=0}^{n-1} \phi'_{t_j} \, \Delta Y_{t_{j+1}}$ converges in some sense to the stochastic integral $\int_0^T \phi'_t \, \mathrm{d}Y_t$ even if the integrator is of infinite variation.
- The continuous-time version of self-financing is $V_T = V_0 + \int_0^T \phi'_t \, \mathrm{d}Y_t$.

From Discrete Time to Continuous Time

- We need adequate tools for modeling asset prices in continuous time that can be interpreted along the lines of

$$(1) \quad \frac{\Delta B_{t+\Delta t}}{B_t} = r \cdot \Delta t$$

$$(2) \quad \frac{\Delta S_{t+\Delta t}}{S_t} = \mu \cdot \Delta t + \sigma \cdot \nu_{t+\Delta t} \cdot \sqrt{\Delta t}$$

  and that preserve the limit distribution of the stock return.
- Replace (1) by an ODE and (2) by an SDE:

$$(1') \quad \frac{\mathrm{d}B_t}{B_t} = r\mathrm{d}t$$

$$(2') \quad \frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

- Replace the self-financing condition $\phi'_{t-\Delta t} Y_t = \phi'_t Y_t$ by $V_T = V_0 + \int_0^T \phi'_t \, \mathrm{d}Y_t$ for an adequately defined stochastic integral.

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Stochastic Processes

- Consider a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$
  - Ω denotes the state space.
  - $\mathcal{A} \subset 2^\Omega$ denotes a sigma algebra that contains all events for which probabilities can be assigned.
  - $(\mathcal{F}_t)_{t\geq 0}$ denotes the filtration, which models the set of information available at time t.
  - $\mathbb{P} : \mathcal{A} \to [0,1]$ is a probability measure, which we refer to as *real-world* probability measure.
- A stochastic process X is a collection of random variables $(X_t)_{t\geq 0}$ indexed by time.
- **Remarks:**
  - Throughout the course, we assume that all processes are continuous (i.e., "no jumps" a.s.) and adapted (i.e., "realization X_t is known at time t"). Formulas become more involved if we relax this assumption.
  - I will avoid technical terms (e.g., measurability, integrability), but focus on economic interpretations. I will rather assume that all processes satisfy all relevant conditions.

Brownian Motion

Definition (Brownian Motion)

A one-dimensional (standard) *Brownian motion* (aka *Wiener Process*) is a stochastic process $W = (W_t)_{t \geq 0}$ such that $W_0 = 0$ a.s. and

- $W_t - W_s \sim \mathcal{N}(0, t - s)$ for $0 \leq s < t$ (*stationary increments*).
- $W_t - W_s$ is independent of $W_u - W_v$ for $0 \leq v < u \leq s < t$ (*independent increments*).

- A k-dimensional standard Brownian motion $W = (W_1, \ldots, W_k)$ is a k-dimensional vector of independent Brownian motions.
- Notice that the paths of a Brownian motion are continuous (a.s.) but nowhere differentiable. In particular, the paths of Brownian motion have infinite length on any interval ("infinite variation").

Brownian Motion

Martingales

Definition (Martingale)

A stochastic process $Z = (Z_t)_{t \geq 0}$ is said to be a *martingale* if "the best estimate of the future is the present", i.e.,

$$E_s[Z_t] = Z_s \qquad t \geq s$$

- Martingales relate to "fair games" and are often thought of as "purely stochastic" processes, that is, containing no trend or being constant in expectation..
- Example: Brownian motion is a martingale.
- There are many generalizations of martingales, e.g.,
  - Submartingales ("non-decreasing in expectation")
  - Supermartingales ("non-increasing in expectation")
  - Local martingales ("if stopped process is a martingale")
  - Semimartingales ("local martingale + process of finite variation")

Itô Integral

- The stochastic integral (a.k.a. Itô integral) is defined by

$$\int_0^T X_t \, \mathrm{d}Z_t = \lim_{n \to \infty} \sum_{j=0}^{n} X_{t_j} (Z_{t_{j+1}} - Z_{t_j})$$

  where Z is a semimartingale, X is an adapted process, and the stochastic limit is taken in the sense of refining partitions (i.e., intermediate points $t_0, t_1, \ldots, t_n$ become more and more dense on the interval $[0, T]$ as n tends to infinity).
- The construction of the limit and prove of convergence is not trivial, since in general the integrator is of infinite variation.
- Such a limit does not necessarily exist pathwise.
- Note: by contrast to the Riemann-Stieltjes integral, the integrand is evaluated at the left end t_j.
- The stochastic integral is itself a random variable.

Stochastic Differential Equation

Definition (Stochastic Differential Equation)

Let W be a standard Brownian motion. An expression of the form

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

for given functions $\mu(t, X_t)$ (*drift*) and $\sigma(t, X_t)$ (*volatility*) is called a *stochastic differential equation* (SDE) driven by Brownian motion and should be understood as a short-hand notation for the integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s.$$

- If the drift $\mu(t, X_t)$ is zero, then the solution is a martingale.
- This definition can be generalized to SDEs driven by jump processes (e.g., Poisson processes).

Quadratic (Co-)Variation

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- Let X, Y be two real-valued stochastic processes, then their *quadratic covariation process* is defined as

$$[X,Y]_t = \lim_{\Delta t \to 0} \sum_{j=0}^{t} (X_{t_{j+1}} - X_j)(Y_{t_{j+1}} - Y_j)$$

- The *quadratic variation process* of X is defined by

$$[X]_t = [X,X]_t$$

- Rules for quadratic (co)-variation:
  - linearity in both arguments
  - $[X,g] = 0$ if g is a continuous function of bounded variation
  - $\mathrm{d}[W_1, W_2] = \rho\,\mathrm{d}t$ for BMs with correlation coefficient ρ; $\mathrm{d}[W] = \mathrm{d}t$
  - if $\mathrm{d}X = \mu_X\,\mathrm{d}t + \sigma_X\,\mathrm{d}W_1$ and $\mathrm{d}Y = \mu_Y\,\mathrm{d}t + \sigma_Y\,\mathrm{d}W_2$, then

$$\mathrm{d}[X,Y] = \sigma_X \sigma_Y \rho\,\mathrm{d}t, \qquad \mathrm{d}[X] = \sigma_X^2\,\mathrm{d}t$$

Theorem (Itô's Lemma for continuous semimartingales)

Let X be a continuous real-valued semimartingale, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)\,\mathrm{d}t + \frac{\partial}{\partial x} f(t, X_t)\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t)\,\mathrm{d}[X, X]_t.$$

Theorem (Itô's Lemma for Itô processes)

Let X be an Itô process $\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}W_t$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\mathrm{d}f(t, X_t) = \left[\frac{\partial}{\partial t} f(t, X_t) + \frac{\partial}{\partial x} f(t, X_t)\mu_X + \frac{1}{2}\frac{\partial^2}{\partial x^2} f(t, X_t)\,\sigma_X^2\right]\mathrm{d}t + \frac{\partial}{\partial x} f(t, X_t)\sigma\,\mathrm{d}W_t.$$

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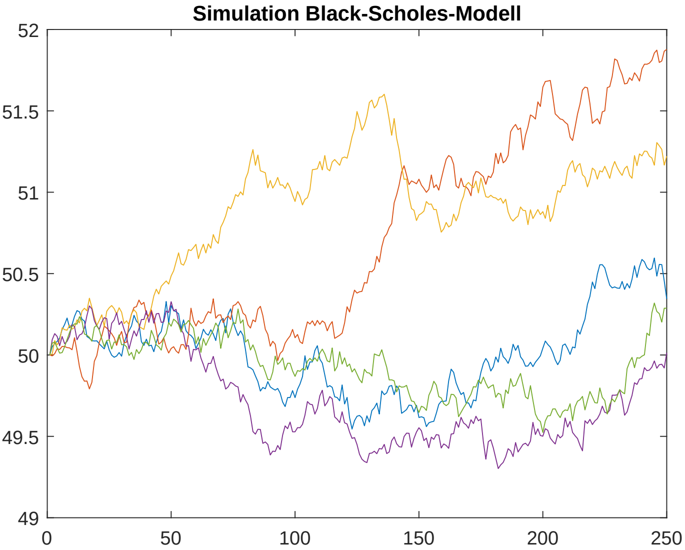
Problem: Derive the stock price in the Black-Scholes model and show that it is strictly positive almost surely.

Solution:

Problem: Black Scholes Model

Geometric Brownian Motion



Itô's Lemma: Multivariate Version

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Theorem (Itô's Lemma for continuous semimartingales)

Let $X = (X_t^1, \ldots, X_t^n)_{t \geq 0}$ be a continuous $\mathbb{R}^n$-valued semimartingale, and $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}$-function, then

$$\begin{aligned} \mathrm{d}f(t, X_t) &= \frac{\partial}{\partial t} f(t, X_t)\, \mathrm{d}t + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(t, X_t)\, \mathrm{d}X_t^i \\ &\quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t)\, \mathrm{d}[X^i, X^j]_t. \end{aligned}$$

Special Case: $f(X, Y) = XY$: Itô product rule:

$$\mathrm{d}(XY)_t = X_t \mathrm{d}Y_t + Y_t \mathrm{d}X_t + \mathrm{d}[X, Y]_t$$

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Theorem (Itô's Lemma for multivariate Itô processes)

Let W be a k-dimensional standard Brownian motion, X be a $\mathbb{R}^n$-valued Itô process with dynamics

$$\mathrm{d}X_t = \mu_X \mathrm{d}t + \sigma_X \mathrm{d}W_t$$

for sufficiently smooth functions $\mu_X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma_X : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$. Let $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,2}$-function with gradient $\nabla f(t, X_t)$ and Hessian matrix $H_f(t, X_t)$, then

$$\mathrm{d}f(t,X_t) = \Big[\underbrace{\frac{\partial}{\partial t} f(t,X_t)}_{\in\mathbb{R}} + \underbrace{\nabla f(t,X_t)}_{\in\mathbb{R}^n} \cdot \underbrace{\mu_X}_{\in\mathbb{R}^n} + \frac{1}{2}\mathrm{tr}\Big(\underbrace{H_f(t,X_t)}_{\in\mathbb{R}^{n\times n}} \underbrace{\sigma_X}_{\in\mathbb{R}^{n\times k}} \underbrace{\sigma_X'}_{\in\mathbb{R}^{k\times n}}\Big)\Big]\mathrm{d}t$$

$$+ \underbrace{\nabla f(t,X_t)}_{\in\mathbb{R}^n} \underbrace{\sigma_X}_{\in\mathbb{R}^{n\times k}} \underbrace{\mathrm{d}W_t}_{\in\mathbb{R}^k}$$

Example: Relative Asset Prices

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Definition (Equivalent Probability Measure)

Two probability measures $\mathbb{P}$ and $\mathbb{Q}$ are said to be *equivalent*, $\mathbb{P} \sim \mathbb{Q}$, if both measures possess the same null sets, i.e., for all events $A \in \mathcal{A}$

$$\mathbb{P}(A) = 0 \quad \Longleftrightarrow \quad \mathbb{Q}(A) = 0.$$

- In our pricing applications, we consider equivalent probability measures that are associated to a numéraire.
- A numéraire is any self-financing portfolio ϕ that generates strictly positive wealth $V_t^\phi = \phi_t' Y_t$
- A probability measure $\mathbb{Q} \sim \mathbb{P}$ is said to be an equivalent martingale measure if for every asset with price process Y^i $(i = 1, \ldots, m)$ the price expressed in terms of the numéraire V_t^ϕ is a martingale under $\mathbb{Q}$.

Change of Measure – Radon-Nikodym Theorem

- The following theorem states how to switch between two equivalent probability measures.

Theorem (Radon-Nikodym)

Let $\mathbb{P} \sim \mathbb{Q}$ denote two equivalent probability measures, then there exists a unique (a.s.), positive random variable $\theta = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ such that

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}[\theta X], \qquad \mathbb{E}^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{Q}}\left[\frac{X}{\theta}\right]$$

for all real-valued random variables X. In particular,

$$\mathbb{Q}[A] = \mathbb{E}^{\mathbb{P}}[\theta\, 1_A]$$

θ is called the *Radon-Nikodym density* (or *Radon-Nikodym derivative*).

- Critical Question: How can we perform a change of measure if the market is driven by Brownian motions?

Theorem (Girsanov)

Suppose that a measure $\mathbb{Q}$ is defined in terms of a measure $\mathbb{P}$ by the Radon-Nikodym process $(\theta_t)_{t\geq 0}$, with

$$\mathrm{d}\theta_t = -\lambda_t \theta_t \,\mathrm{d}W_t$$

where W is a Brownian motion under $\mathbb{P}$ and λ is a continuous adapted process. Then the process $\widetilde{W}$ defined by $\widetilde{W}_0 = 0$ and

$$\mathrm{d}\widetilde{W}_t = \lambda_t \,\mathrm{d}t + \mathrm{d}W_t$$

is a Brownian motion under $\mathbb{Q}$.

This works as well for vector BMs; in this case, write

$$\mathrm{d}\theta_t = -\theta_t \lambda_t' \,\mathrm{d}W_t, \quad \mathrm{d}\widetilde{W}_t = \lambda_t \,\mathrm{d}t + \mathrm{d}W_t.$$

Some Remarks

- The stochastic differential equation $\mathrm{d}\theta_t = -\lambda_t \theta_t \, \mathrm{d}W_t$ has a unique solution, the Radon-Nikodym *process*:

$$\theta_t = \mathcal{E}(\lambda)_t = \exp\left(-\int_0^t \lambda_s \mathrm{d}W_s - \frac{1}{2} \int_0^t \lambda_s^2 \mathrm{d}s \right)$$

- The process $\mathcal{E}(\lambda)$ is called the stochastic exponential or Doléans-Dade exponential of λ.
- The Radon-Nikodym *derivative* is given by

$$\theta_T = \exp\left(-\int_0^T \lambda_s \mathrm{d}W_s - \frac{1}{2} \int_0^T \lambda_s^2 \mathrm{d}s \right)$$

- The Radon-Nikodym *process* is a $\mathbb{P}$-martingale, i.e.,

$$\theta_t = \mathbb{E}_t[\theta_T].$$

Part II

Generic State Space Model

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Generic State Space Model

- We consider a general framework with n state variables and m assets
- The state variables may include asset prices (in this case $X_i = Y_i$) such as
  - Bonds
  - Commodities
  - Money market account
  - Stocks
  - ...
- But they can also model non-tradable financial or economic factors, such as
  - Interest rates
  - Volatility
  - Expected rate of return
  - Inflation
  - GDP growth
  - ...
- The model is driven by k risk sources (Brownian motions).

Generic State Space Model

- General continuous-time financial market model driven by Brownian motion:

Generic State Space Model

$$
\begin{aligned}
\mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\
Y_t &= \pi_Y(t, X_t).
\end{aligned}
$$

- Notation:

$$
\begin{aligned}
W_t &: k\text{-dimensional standard Brownian motion} \\
X_t &: n\text{-dimensional Markov process of state variables} \\
Y_t &: m\text{-dimensional process of asset prices at time } t \\
\mu_X(t, X_t) &: \text{vector of length } n \\
\sigma_X(t, X_t) &: \text{matrix of size } n \times k \\
\pi_Y(t, X_t) &: \text{vector of length } m \\
t &: \text{time, measured in years}
\end{aligned}
$$

Asset Dynamics

- Given the functions μ_X, σ_X, and π_Y, we can determine the asset dynamics $\mathrm{d}Y$ on the basis of Itô's lemma.
- Fix a component $C = Y_i$ ("claim") for some $i = 1, \ldots, m$ from the vector of asset prices $Y = (Y_1, \ldots, Y_m)'$.
- Define the real function $\pi_C = \pi_{Y,i}$. Itô's lemma yields (see slide 31).

$$\mathrm{d}C_t = \mu_C(t, X_t)\,\mathrm{d}t + \sigma_C(t, X_t)\,\mathrm{d}W_t$$

with

$$\begin{aligned}
\mu_C &= \frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X + \frac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X \sigma_X'\Big) \\
&= \frac{\partial \pi_C}{\partial t} + \sum_{i=1}^{n} \frac{\partial \pi_C}{\partial x_i}\mu_{X,i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\ell=1}^{k} \frac{\partial^2 \pi_C}{\partial x_i \partial x_j}\sigma_{X,i,\ell}\sigma_{X,j,\ell} \\
\sigma_C &= \nabla \pi_C\,\sigma_X.
\end{aligned}$$

Example: Black-Scholes Model

- Two assets: money market account M and stock S

$$\begin{aligned} \mathrm{d}S_t &= S_t[\mu \mathrm{d}t + \sigma \mathrm{d}W_t] \\ \mathrm{d}M_t &= M_t r \mathrm{d}t \end{aligned}$$

- This can be written in standard state space form by letting the state variable = asset prices be of dimension $n = m = 2$, with components S_t and M_t.
- There is only one source of uncertainty ($k = 1$).
- The vector functions μ_X, σ_X, and π_Y are given by

$$\mu_X(t, S_t, M_t) = \begin{bmatrix} \mu S_t \\ r M_t \end{bmatrix}, \quad \sigma_X(t, S_t, M_t) = \begin{bmatrix} \sigma S_t \\ 0 \end{bmatrix},$$

$$\pi_Y(t, S_t, M_t) = \begin{bmatrix} S_t \\ M_t \end{bmatrix}.$$

- A *Vasicek process* or *Ornstein-Uhlenbeck process* is a process of the form
$$dX_t = a(b - X_t)\,dt + \sigma\,dW_t.$$
- Properties: X_t fluctuates around the mean-reversion level b. The parameter a determines the mean-reversion speed. We will see later on that this process is normally distributed.
- Vasicek processes are commonly used to model rates such as interest rates, inflation rates, exchange rates, (expected) growth rates, etc.
- The Vasicek process has the (dis-)advantage that it can take positive *and* negative values.
- A prominent alternative is the *Cox-Ingersoll-Ross process*
$$dX_t = a(b - X_t)\,dt + \sigma\sqrt{X_t}\,dW_t,$$
which can only take positive values, but has a very complicated distribution (non-central χ^2).

Model with Stochastic Interest Rates

- The short rate follows a Vasicek process:

$$\begin{aligned} dS_t &= \mu S_t \, dt + \sigma_S S_t \, dW_{1,t} \\ dM_t &= r_t M_t \, dt \\ dr_t &= a(b - r_t) \, dt + \sigma_r \, d\left(\rho W_{1,t} + \sqrt{1-\rho^2} \, W_{2,t}\right). \end{aligned}$$

- $n = 3$ state variables, S_t, M_t, r_t, along with $k = 2$ sources of risk, and $m = 2$ assets S_t, M_t. Vector/matrix functions:

$$\mu_X(t, S_t, M_t, r_t) = \begin{bmatrix} \mu S_t \\ r_t M_t \\ a(b - r_t) \end{bmatrix},$$

$$\sigma_X(t, S_t, M_t, r_t) = \begin{bmatrix} \sigma_S S_t & 0 \\ 0 & 0 \\ \sigma_r \rho & \sigma_r \sqrt{1-\rho^2} \end{bmatrix}, \quad \pi_Y(t, S_t, M_t, r_t) = \begin{bmatrix} S_t \\ M_t \end{bmatrix}.$$

Positive Prices

- If the asset i has a positive price, i.e., π_C maps to the positive real numbers, we can rewrite

$$\begin{aligned} \mathrm{d}C_t &= \mu_C(t, X_t)\,\mathrm{d}t + \sigma_C(t, X_t)\,\mathrm{d}W_t \\ &= C_t\left[\widetilde{\mu}_C(t, X_t)\,\mathrm{d}t + \widetilde{\sigma}_C(t, X_t)\,\mathrm{d}W_t\right] \end{aligned}$$

  with $\widetilde{\mu}_C = \frac{\mu_C}{C}$, $\widetilde{\sigma}_C = \frac{\sigma_C}{C}$.

- Applying Itô's lemma to determine log return:

$$\begin{aligned} \mathrm{d}\log(C) &= C^{-1}\mathrm{d}C + \tfrac{1}{2}(-C^{-2})\mathrm{d}[C] \\ &= \widetilde{\mu}_C\,\mathrm{d}t + \widetilde{\sigma}_C\,\mathrm{d}W_t - \tfrac{1}{2}\widetilde{\sigma}_C\widetilde{\sigma}_C'\,\mathrm{d}t \end{aligned}$$

- Consequently,

$$\log(C_t) = \log(C_0) + \int_0^t (\widetilde{\mu}_C - \tfrac{1}{2}\widetilde{\sigma}_C\widetilde{\sigma}_C')\mathrm{d}s + \int_0^t \widetilde{\sigma}_C\,\mathrm{d}W_s$$

$$\Longrightarrow \quad C_t = C_0 \exp\left(\int_0^t (\widetilde{\mu}_C - \tfrac{1}{2}\widetilde{\sigma}_C\widetilde{\sigma}_C')\mathrm{d}s + \int_0^t \widetilde{\sigma}_C\,\mathrm{d}W_s\right) > 0$$

Self-financing Portfolios

- ϕ_t is the vector of number of units of assets held at time t.
- Portfolio value generated by the *portfolio strategy* ϕ:

$$V_t = \phi_t' Y_t.$$

- A portfolio strategy ϕ is *self-financing* if portfolio rebalancing neither generates nor destroys money, i.e.,

$$\mathrm{d}V_t = \phi_t' \, \mathrm{d}Y_t$$

or equivalently, $V_T = V_0 + \int_0^T \phi_t' \, \mathrm{d}Y_t$. This is the self-financing condition for continuous trading.

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Checking if a Market is Free of Arbitrage

- We consider our generic state space market model

$$\mathrm{d}X_t = \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t$$
$$Y_t = \pi_Y(t, X_t).$$

- A natural question is whether there is an easy-to-check criterion on whether a market satisfies "nice" economic properties.
- Two fundamental economic properties are
  - absence of arbitrage ("no free profits without risk")
  - completeness ("all risks are hedgeble")
- Since the model is formulated in terms of the functions $\mu_X(t, X_t)$, $\sigma_X(t, X_t)$, and $\pi_Y(t, X_t)$, it should be possible to relate these conditions to these functions.

Arbitrage Opportunity

Definition (Arbitrage Opportunity)

1. A self-financing trading strategy ϕ is said to be an arbitrage opportunity if the value V generated by ϕ satisfies the following conditions:

Arb 1.)	$V_0 = 0$	Zero net investment
Arb 2.)	$\mathbb{P}(V_T \geq 0) = 1$	Riskless investment
Arb 3.)	$\mathbb{P}(V_T > 0) > 0$	Chance of making profits

2. A market model is called free of arbitrage if no arbitrage opportunities exist.

"An arbitrage opportunity makes something out of nothing."

Working with a Numéraire

- Asset prices are expressed in terms of a chosen currency (euro, dollar, ...). For theoretical purposes it is often useful to work with a *numéraire*, and to consider *relative* asset price processes (i.e., relative to the numéraire).
- A numéraire N_t is any *asset* (or more generally a self-financing portfolio) whose price is always *strictly positive*, i.e., it has a representation

$$\begin{aligned} dN_t &= \mu_N(t, X_t)dt + \sigma_N(t, X_t)dW_t \\ &= N_t[\widetilde{\mu}_N(t, X_t)dt + \widetilde{\sigma}_N(t, X_t)dW_t] \end{aligned}$$

- A portfolio strategy ϕ_t is self-financing if and only if $d(V_t/N_t) = \phi_t' d(Y_t/N_t)$. The relative value process is then given by

$$\frac{V_t}{N_t} = \frac{V_0}{N_0} + \int_0^t \phi_s' \, d\left(\frac{Y_s}{N_s}\right).$$

- Given: joint process of asset prices $(Y_t)_{t\geq 0}$, and a numéraire $(N_t)_{t\geq 0}$.

First Fundamental Theorem of Asset Pricing

The following are equivalent:

1. The market is free of arbitrage.
2. There is a probability measure $\mathbb{Q}_N \sim \mathbb{P}$ such that $(Y_t/N_t)_{t\geq 0}$ is a martingale under $\mathbb{Q}_N$.

- The measure $\mathbb{Q}_N$ is called an *equivalent martingale measure* (EMM) that corresponds to the numéraire N.
- The direction (2) $\Longrightarrow$ (1) can be proven easily. However, it is a hard task to prove (1) $\Longrightarrow$ (2), because one has to construct an EMM (see Delbean and Schachermayer 2006, *The Mathematics of Arbitrage*) .

Proof of the Easy Part

Proof of the Easy Part (cont'd)

Proposition (No Arbitrage Criterion)

The generic state space model

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t,X_t)\,\mathrm{d}t + \sigma_X(t,X_t)\,\mathrm{d}W_t, \qquad Y_t = \pi_Y(t,X_t), \\ \mathrm{d}Y_t &= \mu_Y(t,X_t)\,\mathrm{d}t + \sigma_Y(t,X_t)\,\mathrm{d}W_t \end{aligned}$$

is free of arbitrage if and only if for all t and x there exists a scalar $r(t,x) \in \mathbb{R}$ and a vector $\lambda(t,x) \in \mathbb{R}^k$ such that

$$\mu_Y(t,x) - r(t,x)\pi_Y(t,x) = \sigma_Y(t,x)\lambda(t,x).$$

Another way to write the equation above:

$$\underbrace{\begin{bmatrix}\sigma_Y & \pi_Y\end{bmatrix}}_{\in\mathbb{R}^{m\times(k+1)}} \underbrace{\begin{bmatrix}\lambda \\ r\end{bmatrix}}_{\in\mathbb{R}^{k+1}} = \underbrace{\mu_Y}_{\in\mathbb{R}^m}$$

Typical Situations

- A *sufficient* condition for absence of arbitrage is that the matrix $[\sigma_Y(t,x) \quad \pi_Y(t,x)]$ is invertible for all t and x. Under this condition, the solution is moreover unique.
- The size of the matrix $[\sigma_Y(t,x) \quad \pi_Y(t,x)]$ is $m \times (k+1)$, where m is the number of assets and k is the number of Brownian motions in the model. So, for the matrix to be invertible, we need

$$m = k + 1$$

  (the number of assets exceeds the number of risk factors by one).
- If $m < k + 1$, typically absence of arbitrage holds, but the solution is not unique. If $m > k + 1$, then special conditions must be satisfied to prevent arbitrage.

Money Market Account I

- Notice that on every arbitrage-free market, there exists a short-term interest rate $r_t = r(t, X_t)$ (*short rate*).

- The natural numéraire (the one that is used if there is no specific reason to choose another one) is the *money market account* which is *locally risk-free* and defined by

$$\mathrm{d}M_t = r_t M_t \,\mathrm{d}t$$

- The money market account evolves according to

$$M_t = M_0 \exp\left(\int_0^t r_s \,\mathrm{d}s\right)$$

- Oftentimes, M is already specified in the dynamics of Y.

Money Market Account II

- If the market is free of arbitrage, but M is *not* a component of Y, one can equip the market with a money market account by enlarging the price vector $\widetilde{\pi}_Y = [\pi_Y \;\; M]'$.
- The extended market is free of arbitrage and pins down the term r in the NA criterion. The following equation is trivially satisfied:

$$\begin{bmatrix}\sigma_M & \pi_M\end{bmatrix}\begin{bmatrix}\lambda \\ r\end{bmatrix} = \mu_M$$

- If the solution for r is unique (but not necessarily the solution for λ), one can indeed construct the money market account, i.e., construct a self-financing portfolio s.t. $\phi' Y = M$.
- **Moral**: Every arbitrage-free market can be equipped with an MMA such that the extended market is still free of arbitrage. Thus, the MMA can be used as a numéraire in any arbitrage-free market.

Market Price of Risk and Risk-neutral Measure

- The process $\lambda_t = \lambda(t, X_t)$ is called the *market price of risk*.
- Given the market price of risk, we can apply Girsanov's theorem and define the Girsanov kernel

$$\theta_t = \mathcal{E}(\lambda)_t = \exp\left(- \int_0^t \lambda_s' \mathrm{d}W_s - \frac{1}{2} \int_0^t \|\lambda_s\|^2 \mathrm{d}s \right)$$

- Then the process $W^{\mathbb{Q}}$ with

$$\mathrm{d}W_t^{\mathbb{Q}} = \lambda_t \, \mathrm{d}t + \mathrm{d}W_t$$

  is a k-dimensional Brownian motion under $\mathbb{Q} \sim \mathbb{P}$.
- **Remark:** This measure $\mathbb{Q} = \mathbb{Q}_M$ is an equivalent martingale measure corresponding to the money market account as numéraire (see slide 72), a so-called *risk-neutral probability measure*.
- **Remark:** Under $\mathbb{Q}$ every traded asset has a drift rate of $r_t = r(t, X_t)$

Proof of the NA Criterion

- The condition for absence of arbitrage in the generic state space model can be written briefly as: there must exist $r = r(t, x)$ and $\lambda = \lambda(t, x)$ such that

$$\mu_Y - r\pi_Y = \sigma_Y \lambda.$$

- We will derive this from the *First Fundamental Theorem of Asset Pricing*. The following concepts will be used:
  - numéraire
  - money market account
  - equivalent martingale measure (EMM)

Proof of the NA Criterion

- Let $\mathbb{Q}_N$ denote a probability measure defined by the RN process λ_N. $\mathbb{Q}_N$ is an EMM if and only if the relative asset price process Y_t/N_t is a $\mathbb{Q}_N$-martingale, i.e., its drift rate under $\mathbb{Q}_N$ is zero.
- The relative asset price process follows

$$\mathrm{d}(Y/N) = \mu_{Y/N}\,\mathrm{d}t + \sigma_{Y/N}\,\mathrm{d}W.$$

- According to Girsanov's Theorem

$$\mathrm{d}\widetilde{W}_t = \lambda_N(t, X_t)\,\mathrm{d}t + \mathrm{d}W_t$$

is a Brownian motion under $\mathbb{Q}_N$. Therefore,

$$\mathrm{d}(Y/N) = \mu_{Y/N}\,\mathrm{d}t + \sigma_{Y/N}\,(\mathrm{d}\widetilde{W}_t - \lambda_N\mathrm{d}t).$$

- Thus, Y/N is a $\mathbb{Q}_N$-martingale if and only if $\mu_{Y/N} = \sigma_{Y/N}\lambda_N$.

Proof of the NA Criterion (cont'd)

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- Choose $N_t = M_t$ (money market account) and write $\lambda_M = \lambda$.
- From $\mathrm{d}M_t = r_t M_t \,\mathrm{d}t$ it follows that

$$\mathrm{d}(M_t^{-1}) = -r_t M_t^{-1} \,\mathrm{d}t.$$

- Therefore by the stochastic product rule,

$$\mathrm{d}(Y/M) = Y \,\mathrm{d}(M^{-1}) + M^{-1} \,\mathrm{d}Y = M^{-1}\big(\mathrm{d}Y - rY \,\mathrm{d}t\big)$$

so that

$$\mu_{Y/M} = M^{-1}(\mu_Y - r\pi_Y), \qquad \sigma_{Y/M} = M^{-1}\sigma_Y.$$

- Because M^{-1} is never zero, the condition $\mu_{Y/M} = \sigma_{Y/M}\lambda$ is equivalent to the *no-arbitrage criterion*

$$\mu_Y - r\pi_Y = \sigma_Y \lambda.$$

Example: Black-Scholes Model

- Asset dynamics

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \qquad dM_t = rM_t \, dt.$$

- The no-arbitrage criterion $\mu_Y - r\pi_Y = \sigma_Y \lambda$ becomes

$$\begin{bmatrix} \mu S \\ rM \end{bmatrix} - r \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} \sigma S \\ 0 \end{bmatrix} \lambda$$

where the quantities that are to be determined are indicated in blue.

- There is a (unique) solution, i.e., the BS model is free of arbitrage (and complete):

$$r = r, \quad \lambda = \frac{\mu - r}{\sigma}$$

- The $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ is given by $W_t^{\mathbb{Q}} = \lambda t + W_t$. Hence, the dynamics under $\mathbb{Q}$ are

$$dS_t = rS_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}}, \qquad dM_t = rM_t \, dt.$$

Example: Model with Stochastic Interest Rates

- The short rate follows the Vasicek model:

$$
\begin{aligned}
\mathrm{d}M_t &= r_t M_t \,\mathrm{d}t \\
\mathrm{d}S_t &= \mu S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}W_{1,t} \\
\mathrm{d}r_t &= a(b - r_t)\,\mathrm{d}t + \sigma_r \,\mathrm{d}\big(\rho W_{1,t} + \sqrt{1-\rho^2}\, W_{2,t}\big).
\end{aligned}
$$

- No-arbitrage criterion

$$
\begin{bmatrix} \mu S \\ rM \end{bmatrix} - r \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} \sigma_S S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}
$$

- There is a (non-unique) solution. The model is free of arbitrage.
- The solution is non-unique because λ_2 is arbitrary. The quantities r and λ_1 are defined uniquely by absence of arbitrage.

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(a) For a given numéraire N, derive the dynamics of Y/N.

(b) Show how the result from (a) simplifies if one chooses $N = M$.

Solution:

Problem: Working with a Numéraire

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The Pricing Problem

- Let an arbitrage-free model be given in the generic state space form

$$\begin{aligned} \mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t, \\ \mathrm{d}Y_t &= \mu_Y(t, X_t)\,\mathrm{d}t + \sigma_Y(t, X_t)\,\mathrm{d}W_t, \qquad Y_t = \pi_Y(t, X_t) \end{aligned}$$

  s.t.

$$\mu_Y(t, X_t) - r(t, X_t)\pi_Y(t, X_t) = \sigma_Y(t, X_t)\lambda(t, X_t).$$

- Suppose now that a new asset is introduced, for instance a contract that will produce a state-dependent payoff at a given time $T > 0$. Pricing on the basis of absence of arbitrage means: the new asset should be priced such that no arbitrage is introduced.
- We want to turn this principle into a *pricing formula*.

Pricing Formula

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- If there is an EMM $\mathbb{Q}_N$, for a given numéraire N_t, the relative price of *any* asset must be a martingale under $\mathbb{Q}_N$. By the martingale property, we therefore have:

Numéraire-dependent pricing formula

Let C_T denote the terminal payoff of a contingent claim that matures at time T. For every EMM $\mathbb{Q}_N$ for a given numéraire N_t, an arbitrage-free price at time t is given by

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right].$$

- This can be used as a *pricing formula* for derivative contracts.
- **Crucial question**: When is the arbitrage-free price of the derivative unique?

Unique Asset Prices

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- To have uniquely defined prices of derivatives, the equation

$$\mu_Y(t,x) - r(t,x)\pi_Y(t,x) = \sigma_Y(t,x)\lambda(t,x)$$

  needs to have a *unique* solution $\big(r(t,x), \lambda(t,x)\big)$. Then the corresponding EMM and the corresponding SDF are uniquely determined.
- One can show that the solution is unique if and only if the matrix $[\pi_Y \quad \sigma_Y]$ has full column rank for all (t,x).
  - Sufficient condition: the matrix $[\pi_Y \quad \sigma_Y]$ is invertible (requires $m = k+1$).
  - Necessary condition: $m \geq k+1$
- In arbitrage-free markets with unique EMM $\mathbb{Q}_N$, the arbitrage-free price $C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]$ is uniquely determined.
- We will see later on that uniqueness of the EMM corresponds to an important economic property: *market completeness*.

Verification of Absence of Arbitrage

- The process C_t is defined by

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]$$

where C_T is a given random variable.

- In applications, the terminal payoff of the derivative, C_T, is a function of the state vector at time T: $C_T = F(X_T)$.
- To ensure that no arbitrage is introduced by the price process C_t, we need to verify that the process $(C_t/N_t)_{t\geq 0}$ is a martingale; i.e., the martingale property holds for *any* s and t with $s < t$, not just for t and T.
- This follows from the tower law of conditional expectations:

$$E_s^{\mathbb{Q}_N}\left[\frac{C_t}{N_t}\right] = E_s^{\mathbb{Q}_N}\left[E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]\right] = E_s^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right] = \frac{C_s}{N_s}\,.$$

Money Market Account as a Numéraire

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- In principle, every self-financing portfolio which generates positive wealth can act as a numéraire.
- However, there are several commonly used choices:
    - Money market account
    - Stock
    - Numéraire portfolio
    - ...
- Using the money market account as a numéraire, the pricing formula becomes

$$C_t = B_t E_t^{\mathbb{Q}}\left[\frac{C_T}{B_T}\right] = E_t^{\mathbb{Q}}\left[C_T \frac{B_t}{B_T}\right] = E_t^{\mathbb{Q}}\left[C_T \, \mathrm{e}^{-\int_t^T r_s ds}\right]$$

- We refer to $\mathbb{Q} = \mathbb{Q}_M$ as the *risk-neutral pricing measure*. Under $\mathbb{Q}$, the agent discounts at the risk-free rate and does not require a risk premium.
- Under $\mathbb{Q}$ every *traded asset* has an expected return of $r = r(t, X_t)$.

The Numéraire Portfolio

- **Natural question**: Is there a numéraire N for which $\mathbb{Q}_N = \mathbb{P}$?
- In an arbitrage free market driven by Brownian motion, one can show that the answer is positive if one can solve the problem of maximizing expected log-utility from terminal wealth, i.e., if

$$\max_{\phi} \mathbb{E}[\log(V_T^{\phi})] < \infty$$

- The portfolio ρ that maximizes this optimization problem will be called the *log-optimal portfolio* or the *numéraire portfolio*.
- One can show that using the numéraire portfolio as numéraire N, the pricing formula becomes the *real-world pricing formula*

$$C_t = \mathbb{E}_t\left[C_T \frac{V_t^{\rho}}{V_T^{\rho}}\right]$$

where the expectation is calculated under $\mathbb{P}$.

Alternative Formulation of the FTAP

- Instead of exploiting an eqivalent martingale measure, it is also very common to make use of a *stochastic discount factor* (SDF) or *pricing kernel*.
- A stochastic discount factor K is a positive adapted process with $K_0 = 1$ such that the process $(K_t\, Y_t)$ is a martingale under $\mathbb{P}$, i.e.,

$$\mathbb{E}_t[K_s Y_s] = K_t Y_t$$

- One can show that the existence of an EMM is equivalent to the existence of a SDF. Therefore, the FTAP can also be formulated in terms of the SDF:

First Fundamental Theorem of Asset Pricing

The following are equivalent:

1. The market is free of arbitrage.
2. There is a stochastic discount factor.

Some Properties of the SDF

- The SDF is a positive adapted process, i.e., it can be written as (see slide 46)

$$K_t = \exp\left(\int_0^t (\widetilde{\mu}_K - \tfrac{1}{2}\widetilde{\sigma}_K\widetilde{\sigma}_K')\mathrm{d}s + \int_0^t \widetilde{\sigma}_K \,\mathrm{d}W_s\right)$$

- By definition of the SDF, the process $KM = (K_t M_t)_{t\geq 0}$ must be a martingale under $\mathbb{P}$. It follows from Itô's lemma that

$$\mathrm{d}(KM)_t = K_t M_t[(r + \widetilde{\mu}_K)\mathrm{d}t + \widetilde{\sigma}_K \mathrm{d}W_t]$$

  where $\widetilde{\sigma}_K = -\lambda'$. The martingale property implies $\widetilde{\mu}_K = -r$.
- The SDF combines the role of discounting at the short rate and the change of measure from $\mathbb{P}$ to $\mathbb{Q}$.
- It follows that the numéraire portfolio and the pricing kernel are inversely related, i.e., $K_t = \frac{1}{V_t^{\rho}}$.

Multiple Payoffs

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- A contract may generate payoffs (possible uncertain) at multiple points in time.
- Such a contract can be viewed as a portfolio of options with individual payoff dates. The value of the portfolio is the sum of the values of its constituent parts.
- We get, for a contract with payoffs $\hat{C}_{T_i}$ at times T_i $(i = 1, \ldots, n)$:

$$C_0 = N_0 \sum_{i=1}^{n} E^{\mathbb{Q}_N}\left[\frac{\hat{C}_{T_i}}{N_{T_i}}\right].$$

- In the special case of constant interest rates, we can take the money market account $M_t = e^{rt}$ as the numéraire; then

$$C_0 = \sum_{i=1}^{n} e^{-rT_i} E^{\mathbb{Q}}[\hat{C}_{T_i}].$$

- This shows that the NDPF can be seen as a generalized *net present value formula*.

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Replication

- So far, we have talked about no-arbitrage and uniqueness of arbitrage-free prices. We now turn to the natural question of whether we can hedge risks and replicate payoffs.
- Let an arbitrage-free model be given in the generic state space form

$$
\begin{aligned}
\mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t, \\
\mathrm{d}Y_t &= \mu_Y(t, X_t)\,\mathrm{d}t + \sigma_Y(t, X_t)\,\mathrm{d}W_t, \qquad Y_t = \pi_Y(t, X_t)
\end{aligned}
$$

s.t.

$$
\mu_Y(t, X_t) - r(t, X_t)\pi_Y(t, X_t) = \sigma_Y(t, X_t)\lambda(t, X_t).
$$

- If we want to price a claim, a natural question is whether this derivative can be replicated by a self-financing trading strategy ϕ.

Replication and Complete Market

Definition (Replication Strategy, Completeness)

Let $C_T = F(X_T)$ be the terminal payoff of a contingent claim.

1. A self-financing portfolio strategy ϕ is called a *replication strategy* or *hedging strategy* for C if

$$V_T^{\phi} = C_T$$

2. The claim is said to be *attainable* if there exists a replication strategy ϕ for this claim.
3. A market is said to be complete if and only if every claim is attainable.

- A replication strategy is thus a portfolio whose value is, under all circumstances, equal to the value of a specified contingent claim.
- Market completeness is a desirable property but typically not met in reality.

Pricing by Replication

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Lemma (Law of One Price)

Suppose the market is arbitrage-free.

1. For an attainable contingent claim C with hedging strategy ϕ,

$$C_0 = V_0^{\phi}$$

is the unique arbitrage-free price, i.e., trading in the primary assets *and* the derivative does not lead to arbitrage opportunities.

2. If $V_T^{\phi} = V_T^{\psi}$ for trading strategies ϕ and ψ, then

$$V_0^{\phi} = V_0^{\psi}.$$

- The proof is trivial and does not rely on specific asset dynamics.

When is Replication Possible?

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- We need an easy-to-check criterion when replication is possible.

Second Fundamental Theorem of Asset Pricing

For an arbitrage-free market, the following are equivalent:

1. The market is complete.
2. For any given numéraire N, the corresponding EMM $\mathbb{Q}_N \sim \mathbb{P}$ is unique.

- We have already seen that for an arbitrage-free market, the EMM is unique if and only if the matrix $[\pi_Y(t,x)\ \sigma_Y(t,x)] \in \mathbb{R}^{m \times (k+1)}$ has full column rank for all (t,x).
- Consequently, if there are enough traded assets ($m > k+1$ is necessary) in the model to determine prices uniquely, then they are also enough to make replication possible. And vice versa.

Examples

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- Obviously, the Black Scholes model (see slides 42, 63) is complete since
$$[\pi_Y \; \sigma_Y] = \begin{bmatrix} S_t & S_t\sigma \\ M_t & 0 \end{bmatrix}$$
is invertible for every combination of S_t and $M_t > 0$. Besides, there was a unique solution for r and λ, which uniquely determines the change of measure.

  $\Longrightarrow$ Pricing by replication is always possible.

- The model with stochastic interest rates of the Vasicek type (see slides 45, 64) is incomplete ($m = k = 2$), and the EMM is not unique since there is no unique solution for λ_2.

  $\Longrightarrow$ Pricing by replication is in general impossible.

  However, the model can be completed by adding a bond that can be used to hedge interest rate risk (see Chapter 6).

The Replication Recipe

To replicate a payoff at time T given by $C_T = F(X_T)$, we follow a four-step procedure:

Step 1. Choose a numéraire N_t and determine the function

$$\pi_C(t,x) = \pi_N(t,x)\, E^{\mathbb{Q}_N}\Big[\frac{F(X_T)}{\pi_N(T,X_T)}\,\Big|\, X_t = x\Big].$$

Step 2. Compute $\sigma_C(t,x) = \nabla\pi_C(t,x)\,\sigma_X(t,x)$.

Step 3. Solve for $\phi = \phi(t,x)$ from

$$[\sigma_C \quad \pi_C] = \phi'[\sigma_Y \quad \pi_Y].$$

Step 4. Start with initial capital $\pi_C(0,X_0)$, and rebalance your portfolio along the trading strategy $\phi_t = \phi(t,X_t)$.

Validity of the replication recipe

- To show the validity of the replication recipe, three conditions need to be demonstrated:

  (i) the equation $[\sigma_C \quad \pi_C] = \phi'[\sigma_Y \quad \pi_Y]$ (where ϕ is the unknown) can be solved

  (ii) the portfolio value generated by the trading strategy ϕ at time T is equal to $V_T^\phi = F(X_T)$.

  (iii) the trading strategy $\phi_t = \phi(t, X_t)$ is self-financing

- These items will be discussed on the next slides.

Property of the function π_C

- We already know that the process defined by $C_t = \pi_C(t, X_t)$ with

$$\pi_C(t,x) = \pi_N(t,x)\,\mathbb{E}_t^{\mathbb{Q}_N}\left[\frac{F(X_T)}{\pi_N(T,X_T)}\right]$$

  is such that C_t/N_t is a martingale under $\mathbb{Q}_N$.
- This property is translated into state space terms as follows: let $r = r(t,x)$ and $\lambda = \lambda(t,x)$ be defined as the solution of the equation (NA criterion):

$$\mu_Y - r\pi_Y = \sigma_Y \lambda.$$

- Then we also have

$$\mu_C - r\pi_C = \sigma_C \lambda.$$

Requirement (i)

- Market completeness means that the EMM for any given numéraire is uniquely defined, i.e., the equation

$$\underbrace{\mu_Y}_{\in\mathbb{R}^m} = \underbrace{\begin{bmatrix}\sigma_Y & \pi_Y\end{bmatrix}}_{\in\mathbb{R}^{m\times(k+1)}} \underbrace{\begin{bmatrix}\lambda \\ r\end{bmatrix}}_{\in\mathbb{R}^{k+1}}$$

  has a *unique* solution $[\lambda \;\; r]'$.
- In other words, the matrix $[\sigma_Y \;\; \pi_Y] = [\sigma_Y(t,x) \;\; \pi_Y(t,x)]$ has rank $k+1$ for all t and x (its columns are linearly independent).
- Because row rank $=$ column rank, this implies that the *rows* of the matrix span the $(k+1)$-dimensional space. This means that the equation

$$[\sigma_C \;\; \pi_C] = \phi'[\sigma_Y \;\; \pi_Y].$$

  has a unique solution ϕ. So requirement (i) is indeed satisfied.

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- Define the portfolio strategy $\phi_t = \phi(t, X_t)$. The corresponding portfolio value is $V_t = \phi_t' Y_t$. Because $\phi' \pi_Y = \pi_C$, this implies that $V_t = C_t$ for all t. In particular, $V_T = F(X_T)$ (requirement (ii)).

- Because $\phi' \pi_Y = \pi_C$ and $\phi' \sigma_Y = \sigma_C$, and because $\mu_Y = r\pi_Y + \sigma_Y \lambda$ as well as $\mu_C = r\pi_C + \sigma_C \lambda$, we have

$$\phi' \mu_Y = \phi'(r\pi_Y + \sigma_Y \lambda) = r\phi' \pi_Y + \phi' \sigma_Y \lambda = r\pi_C + \sigma_C \lambda = \mu_C.$$

- Therefore,

$$\mathrm{d}V = \mu_C \, \mathrm{d}t + \sigma_C \, \mathrm{d}W = \phi'(\mu_Y \, \mathrm{d}t + \sigma_Y \, \mathrm{d}W) = \phi' \mathrm{d}Y$$

which shows that the proposed portfolio strategy is self-financing (requirement (iii)).

Example: Call Option in BS Model

- BS model under $\mathbb{Q}$ (check!):

$$\begin{aligned} dS_t &= rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}} \\ dM_t &= rM_t\,dt. \end{aligned}$$

- Payoff at time T: $\max(S_T - K, 0)$.
- Step 1: determine the pricing function:

$$\pi_C(t, S_t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$

with

$$d_{1,2} = \frac{\log(S_t/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Example: Call Option in BS Model (cont'd)

- Step 2: compute

$$\sigma_C(t, S_t) = \frac{\partial \pi_C}{\partial S_t}(t, S_t)\, \sigma S_t = \Phi(d_1)\sigma S_t.$$

- Step 3: solve for $\phi(t, S_t) = [\phi_S(t, S_t) \;\; \phi_M(t, S_t)]$ from

$$[\Phi(d_1)\sigma S_t \quad S_t\Phi(d_1) - e^{-r(T-t)} K\Phi(d_2)] = [\phi_S \quad \phi_M] \begin{bmatrix} \sigma S_t & S_t \\ 0 & M_t \end{bmatrix}.$$

- We find

$$\phi_S(t, S_t) = \Phi(d_1)$$

$$\phi_M(t, S_t) = -K\,\Phi(d_2)$$

Delta Hedging

- The "delta" of an option is the derivative of the option price with respect to the value of the underlying Y_i, i.e.,

$$\Delta_C = \frac{\partial \pi_C}{\partial Y_i}$$

- There could be several underlying assets (for instance in the case of an option written on the maximum of two stocks), and in that case there are also several deltas.
- In models driven by a single Brownian motion, if an option depends on a single underlying asset, then the number of units of the underlying asset that should be held in a replicating portfolio is given by the delta of the option (as in the example). The resulting strategy is called the *delta hedge*.
- Under certain conditions this also works in the case of multiple underlyings.

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The Pricing PDE

- Compute μ_C and σ_C (Itô's lemma):

$$\mu_C = \frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\, \sigma_X \sigma_X'\Big)$$

$$\sigma_C = \nabla \pi_C \sigma_X .$$

- The equation $\mu_C - r\pi_C = \sigma_C \lambda$ becomes:

Pricing PDE

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \underbrace{(\mu_X - \sigma_X \lambda)}_{=\mu_X^{\mathbb{Q}_N}} + \tfrac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\, \sigma_X \sigma_X'\Big) = r\pi_C, \qquad \pi_C(T, x) = F(x)$$

- This is a *partial differential equation* for the pricing function π_C.
- Notice that the *boundary condition* $\pi_C(T, x) = F(x)$ determines the type of the derivative.

Remarks

- In a model without any non-traded state variables, i.e., $Y = X$, $\pi_Y = x$, the NA condition becomes

$$\mu_X - \sigma_X \lambda = r\,x$$

- The PDE collapses to

$$\frac{\partial \pi_C}{\partial t} + r \nabla \pi_C \cdot x + \frac{1}{2} \mathrm{tr}\Big(H_{\pi_C}\, \sigma_X \sigma_X' \Big) = r \pi_C$$

- The drift term of the spatial first-order derivatives is r, which is the drift term of traded assets under $\mathbb{Q}$.
- The PDE may be solved analytically or numerically (finite-difference methods – generalization of tree methods).
- The PDE can also be derived using the Feynman-Kac Theorem: a mathematical statement that connects the theory of partial differential equations to conditional expectations.

Theorem (Feynman-Kac)

Consider the following parabolic partial differential equation

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot \mu_X^{\mathbb{Q}}(t,x) + \frac{1}{2}\mathrm{tr}\Big(H_{\pi_C}\,\sigma_X(t,x)\sigma_X(t,x)'\Big) + f(t,x) = r(t,x)\pi_C$$

subject to the terminal condition $\pi_C(T,x) = F(x)$. Then, the solution can be written as a conditional expectation

$$\pi_C(t,x) = \mathbb{E}^{\mathbb{Q}}_{t,x}\left[\int_t^T e^{-\int_t^s r(\tau,X_\tau)\,d\tau} f(s,X_s)\mathrm{d}s + e^{-\int_t^T r(\tau,X_\tau)\,d\tau} F(X_T)\right]$$

under $\mathbb{Q}$ such that X is an Itô process driven by the equation

$$\mathrm{d}X = \mu_X^{\mathbb{Q}}(t,X)\,\mathrm{d}t + \sigma_X(t,X)\,\mathrm{d}W^{\mathbb{Q}},$$

with $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

Example: Black-Scholes PDE

- Under $\mathbb{Q}$, the dynamics are

$$\begin{aligned} \mathrm{d}S_t &= rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}} \\ \mathrm{d}M_t &= rM_t\,\mathrm{d}t. \end{aligned}$$

- Therefore, the BSPDE for a derivative with terminal payoff $F(S_T)$ reads

$$\frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} Sr + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2} S^2\sigma^2 = r\,\pi_C$$

  s.t. $\pi_C(T, S_T) = F(S_T)$

- In their original paper Black and Scholes (1973), derived this formula using a different approach and made two mistakes which cancel each other out. Merton (1973) corrected these mistakes and came up with the same PDE.
- The PDE can be transformed to the so-called *heat equation*, which is commonly used in physics and has a well-known solution.

Example: Pricing PDE with Stoch. Interest Rates

- Under $\mathbb{Q}$, the dynamics are

$$\begin{aligned} \mathrm{d}M_t &= r_t M_t \,\mathrm{d}t \\ \mathrm{d}S_t &= r_t S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}W_{1,t}^{\mathbb{Q}} \\ \mathrm{d}r_t &= a^{\mathbb{Q}}(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma_r \,\mathrm{d}\big(\rho W_{1,t}^{\mathbb{Q}} + \sqrt{1-\rho^2}\, W_{2,t}^{\mathbb{Q}}\big). \end{aligned}$$

- Notice that the risk-neutral measure is not uniquely determined since the market price of risk $\lambda = (\lambda_1 \ \lambda_2)$ is not unique.
- Therefore, the pricing PDE for a derivative with payoff $F(r_T, S_T)$ reads

$$\begin{aligned} r\,\pi_C = {} & \frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} S r + \frac{\partial \pi_C}{\partial r} a^{\mathbb{Q}}(b^{\mathbb{Q}} - r) \\ & + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2} S^2 \sigma_S^2 + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial r^2}\sigma_r^2 + \frac{\partial^2 \pi_C}{\partial r \partial S}\rho \sigma_r \sigma_S S \end{aligned}$$

s.t. $\pi_C(T, r_T, S_T) = F(r_T, S_T)$

Summary

- Generic State Space Model:

$$dX_t = \mu_X(t, X_t)\,dt + \sigma_X(t, X_t)\,dW_t, \qquad Y_t = \pi_Y(t, X_t)$$

- No-arbitrage condition (from FTAP 1):

$$\mu_Y - r\pi_Y = \sigma_Y \lambda$$

- Numéraire-dependent pricing formula:

$$\frac{C_t}{N_t} = E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right]$$

- Replication recipe (from FTAP 2) if $\mathrm{rk}(\sigma_Y \ \pi_Y) = k + 1$:

$$[\sigma_C \quad \pi_C] = \phi'[\sigma_Y \quad \pi_Y]$$

- Pricing via PDE:

$$\frac{\partial \pi_C}{\partial t} + \nabla \pi_C \cdot (\mu_X - \sigma_X \lambda) + \frac{1}{2}\mathrm{tr}\left(H_{\pi_C}\, \sigma_X \sigma_X'\right) = r\pi_C$$

Part III

Contingent Claim Pricing

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- This chapter studies examples for contingent claim pricing in several tangible specifications of the GSSM.

Option

1. A European option is a contract between two counterparties, whereby the buyer (= holder) has the right to buy (= Call option) or to sell (= Put option) the underlying from/to the seller (= stillholder) for a predetermined strike price K *at* its maturity T.
2. An American option has the feature that the option can be exercised *before* maturity, i.e., in $[0, T]$.

- Option profile at maturity T on a stock with price process S:

$$
\begin{aligned}
C_T &= (S_T - K)^+ = \max\{S_T - K, 0\} \\
P_T &= (K - S_T)^+ = \max\{K - S_T, 0\}
\end{aligned}
$$

Option Profiles

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How to come up with the Black-Scholes Formula?

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- The Black-Scholes formula is probably the most famous formula in quantitative finance and the starting point of modern financial mathematics.
- Black and Scholes (1973) derive the formula by transforming the BSPDE to the heat equation, which has a well-known solution

$$r\,\pi_C = \frac{\partial \pi_C}{\partial t} + \frac{\partial \pi_C}{\partial S} Sr + \frac{1}{2}\frac{\partial^2 \pi_C}{\partial S^2} S^2 \sigma_S^2$$

  s.t. $\pi_C(T, S_T) = \max(S_T - K, 0)$.
- Besides solving the BSPDE, the problem can be tackled by several approaches, e.g.,
  - Pricing under the EMM $\mathbb{Q}$
  - Pricing under $\mathbb{P}$ using the SDF / numéraire portfolio
  - Taking the limit of a sequence of binomial models
  - Splitting the payoff into two parts and tackle them under two different measures
  - ...

Examples: Pricing Approaches

The Fastest Way to the Black-Scholes Formula

- The European call option has payoff function

$$C_T = \max(S_T - K, 0) = 1_{\{S_T \geq K\}}(S_T - K).$$

- The price of the European put option with payoff $P_T = \max(K - S_T, 0)$ can be obtained from the put-call parity

$$P_t = C_t - S_t + K\mathrm{e}^{-r(T-t)}.$$

- We can decompose the call option into two options;
  1. a long position in the *stock-or-nothing option* which has payoff $1_{\{S_T \geq K\}} S_T$
  2. a short position in the *cash-or-nothing option* which has payoff $1_{\{S_T \geq K\}} K$.
- The price of the call option is determined if we know the prices of the stock-or-nothing option and the cash-or-nothing option.

Cash-or-nothing Option under $\mathbb{Q}$

- Cash-or-nothing option, $C_T^m = 1_{\{S_T \geq K\}} K$ will be priced under $\mathbb{Q}$:

$$\frac{C_0^m}{M_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{C_T^m}{M_T}\right] = \frac{K}{M_T}\,\mathbb{E}^{\mathbb{Q}}\left[1_{\{S_T \geq K\}}\right] = \frac{K}{M_T}\,\mathbb{Q}_M(S_T \geq K).$$

- Under $\mathbb{Q}$, the evolution of the stock price is given by

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}},$$

  where $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

- Therefore:

$$S_T = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right), \qquad Z \overset{\mathbb{Q}}{\sim} N(0,1)$$

$$\implies \mathbb{Q}(S_T \geq K) = \Phi(d_2), \quad d_2 = \frac{\log(S_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Stock-or-nothing option under $\mathbb{Q}_S$

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- Stock-or-nothing option, $C_T^s = 1_{\{S_T \geq K\}} S_T$:

$$\frac{C_0^s}{S_0} = E^{\mathbb{Q}_S}\left[\frac{C_T^s}{S_T}\right] = E^{\mathbb{Q}_S}\left[1_{\{S_T \geq K\}}\right] = \mathbb{Q}_S(S_T \geq K).$$

- Under $\mathbb{Q}_S$, the evolution of the stock price is given by

$$\mathrm{d}S_t = (r + \sigma^2) S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}W_t^{\mathbb{Q}_S}$$

where $W^{\mathbb{Q}_S}$ is a Brownian motion under $\mathbb{Q}_S$.

- Therefore:

$$S_T = S_0 \exp\left((r + \tfrac{1}{2}\sigma^2) T + \sigma\sqrt{T} Z\right), \qquad Z \overset{\mathbb{Q}_S}{\sim} N(0,1)$$

$$\implies \mathbb{Q}_S(S_T \geq K) = \Phi(d_1), \quad d_1 = \frac{\log(S_0/K) + (r + \frac{1}{2}\sigma^2) T}{\sigma\sqrt{T}}.$$

- Putting everything together:

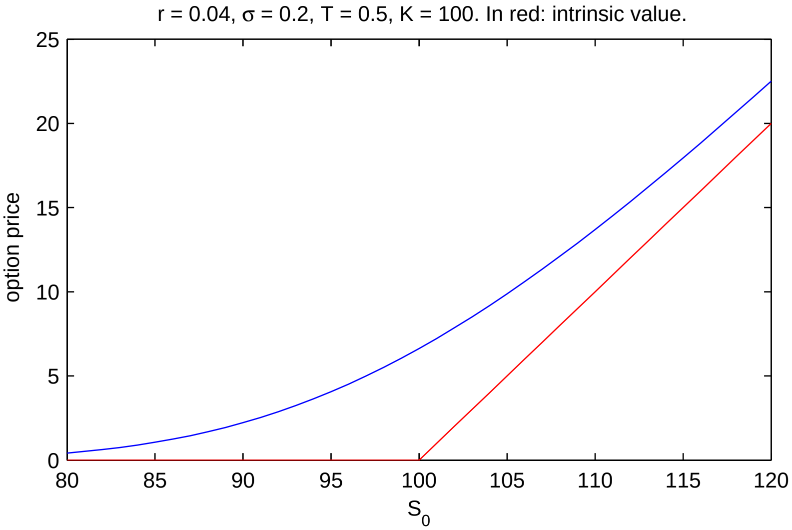
$$C_0 = C_0^s - C_0^m = S_0\Phi(d_1) - e^{-rT}K\Phi(d_2).$$

- The price of the call option is equal to the current value of the stock times the probability under $\mathbb{Q}_S$ that the option will end in the money, minus the current value of the strike times the probability under $\mathbb{Q}$ that the option will end in the money.

Problem: Derive the $\mathbb{Q}_S$-Stock Dynamics

Problem: Derive the $\mathbb{Q}_S$-Stock Dynamics

Option Price versus Intrinsic Value



Critique: Black-Scholes Model

- Volatility, interest rate, expected return are assumed to be constant. $\longrightarrow$ Volatility Smile
- Returns are assumed to be normally distributed. $\longrightarrow$ Underestimation of extreme events.
- Model builds upon a complete market without frictions (no taxes, transaction costs, short-selling constraints, ...).
- Implied volatility $\neq$ historical volatility
  - These caveats become visible if one investigates what volatilities are necessary to explain option prices by the Black-Scholes formula.
  - Implied volatility is not constant, but depends on K and T.
  - If the option is at-the-money, implied volatility is lowest (volatility smile).
- Some of these points can be tackled by adding non-traded state variables to the model.

A Double-barrier Option

- A *perpetual up-and-out down-and-in digital double barrier option* is a contract that is specified by
  - an underlying S_t (for instance a stock index)
  - a lower barrier L
  - an upper barrier U
  - a fixed payoff amount K.
- The contract pays the amount K when the stock price S_t reaches the lower barrier L, but only if the stock price has not reached the upper barrier first. (i.e., the contract "knocks out" when the stock price S_t reaches U.)
- As long as neither the lower nor the upper barrier has been reached, the contract stays alive.
- Therefore the time of expiry of the contract is random (determined in terms of the process S_t).

PDE Approach

- Assume that the BS model holds for the stock price S_t. The Black-Scholes equation for the pricing function $\pi_C(t, S_t)$ is in general

$$\frac{\partial \pi_C}{\partial t}(t, S) + rS\,\frac{\partial \pi_C}{\partial S}(t, S) + \frac{1}{2}\sigma^2 S^2\,\frac{\partial^2 \pi_C}{\partial S^2}(t, S) = r\pi_C(t, S).$$

- Since π_C does not depend on t, this reduces to the ODE

$$rS\,\frac{\mathrm{d}\pi_C}{\mathrm{d}S}(S) + \frac{1}{2}\sigma^2 S^2\,\frac{d^2 \pi_C}{\mathrm{d}S^2}(S) = r\pi_C(S).$$

- Boundary conditions for the up-and-out down-and-in option:

$$\pi_C(U) = 0, \qquad \pi_C(L) = K.$$

Solving the ODE

- We have a linear homogeneous second-order ODE, so the general solution is a linear combination of two particular solutions.
- These solutions should be self-financing portfolios whose values depend only on S_t. One solution is S_t itself (obviously!), another is $S_t^{-\gamma}$ with $\gamma = 2r/\sigma^2$.
- The solution is therefore given by

$$\pi_C(S_t) = c_1 S_t + c_2 S_t^{-\gamma}$$

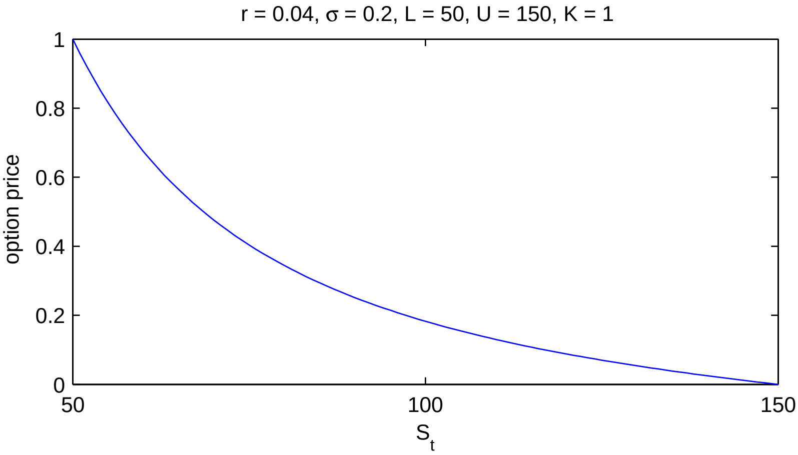
  where the constants c_1 and c_2 should be chosen such that

$$\pi_C(U) = c_1 U + c_2 U^{-\gamma} = 0, \quad \pi_C(L) = c_1 L + c_2 L^{-\gamma} = K.$$

- This linear system has a unique solution.

- Putting everything together yields

$$\pi_C(t, S_t) = \frac{L^\gamma K}{U^{\gamma+1} - L^{\gamma+1}} \left(U^{\gamma+1} S_t^{-\gamma} - S_t \right).$$



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An Example: The Heston Model

- Modeling Stochastic Volatility by a CIR process

$$
\begin{aligned}
dM_t &= M_t r dt \\
dS_t &= S_t[\mu dt + \sqrt{\nu_t} dW_{1,t}] \\
d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} d(\rho W_{1,t} + \sqrt{1-\rho^2} W_{2,t})
\end{aligned}
$$

- The model has five input parameters:
  - ν_0, the initial variance.
  - θ , the mean-reversion variance of the stock price
  - κ, the mean-reversion speed of the variance of the stock price
  - ρ the correlation of the two Wiener processes.
  - σ the volatility of the volatility, or 'vol of vol'
- $n = 3$ state variables, $k = 2$ sources of risk, and $m = 2$ assets:

$$
\mu_X = \begin{bmatrix} \mu S_t \\ r M_t \\ \kappa(\theta - \nu_t) \end{bmatrix}, \quad \sigma_X = \begin{bmatrix} \sqrt{\nu_t} S_t & 0 \\ 0 & 0 \\ \sigma\rho\sqrt{\nu_t} & \sigma\sqrt{\nu_t}\sqrt{1-\rho^2} \end{bmatrix}, \quad \pi_Y = \begin{bmatrix} S_t \\ M_t \end{bmatrix}
$$

Economic Properties

- The model is free of arbitrage: The NA criterion $\mu_Y - \pi_Y r = \sigma_Y \lambda$ yields

$$\begin{bmatrix} \mu S \\ rM \end{bmatrix} - r \begin{bmatrix} S \\ M \end{bmatrix} = \begin{bmatrix} \sqrt{\nu_t} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

- The market price of stock risk is uniquely determined, $\lambda_1 = \frac{\mu - r}{\sqrt{\nu_t}}$.
- The market price of volatility risk λ_2 can be chosen arbitrarily.
- The model is obviously incomplete. Thus for any given numéraire, the corresponding EMM is not unique.
- Consequently, neither the numéraire-dependent option pricing formula, nor the PDE approach deliver unique arbitrage-free option prices. They rather depend on the particular choice of λ_2.

Change of Measure

- Under $\mathbb{Q}$, generated by $(\lambda_1 \ \lambda_2)$, the model evolves according to

$$d S_t = S_t[r dt + \sqrt{\nu_t} dW_{1,t}^{\mathbb{Q}}]$$

$$d\nu_t = \Big[\kappa(\theta - \nu_t) - \underbrace{\lambda_{1,t}\sigma\rho\sqrt{\nu_t}}_{=(\mu - r)\sigma\rho} - \underbrace{\lambda_{2,t}\sigma\sqrt{\nu_t}\sqrt{1-\rho^2}\Big]}\, dt$$

$$+ \sigma\sqrt{\nu_t} d(\rho W_{1,t}^{\mathbb{Q}} + \sqrt{1-\rho^2} W_{2,t}^{\mathbb{Q}})$$

- Heston (1993) *chooses* $\lambda_{2,t}$ such that the drift adjustment is proportional to ν_t, i.e., $\lambda \nu_t$ for $\lambda \in \mathbb{R}$
- Therefore,

$$d\nu_t = \big[\kappa(\theta - \nu_t) - \lambda\nu_t\big] dt + \sigma\sqrt{\nu_t} d(\rho W_{1,t}^{\mathbb{Q}} + \sqrt{1-\rho^2} W_{2,t}^{\mathbb{Q}})$$

and there is a closed-form solution for the call option price for every particular choice of $\lambda \in \mathbb{R}$.

Problem: Set up the Pricing PDE

Calibration vs. Estimation

- **Crucial Question:** How do we determine the market price of risk?
- Calibration and estimation are two ways of determining parameters in a financial model. The difference is:
  - estimation uses methods of statistics/econometrics to infer parameter values from observed *historical* behavior of asset prices and other relevant quantities
  - calibration sets parameter values so as to generate a close match between derivative prices obtained from the model and prices observed *currently* in the market.
- Estimation comes with standard errors, significance tests, and so on; analogous quantities that may serve as warning signals are not produced by calibration.
- Estimation works with models that are written under $\mathbb{P}$ (real-world measure); calibration can be applied to models that are written under $\mathbb{Q}_N$ (martingale measure corresponding to a chosen numéraire).

In our Situation

- Estimation helps us to figure out the parameters under $\mathbb{P}$

$$\begin{aligned} dM_t &= M_t r dt \\ dS_t &= S_t[\mu dt + \sqrt{\nu_t} dW_{1,t}] \\ d\nu_t &= \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t} d(\rho W_{1,t} + \sqrt{1-\rho^2} W_{2,t}) \end{aligned}$$

- However, for pricing purposes, we need the $\mathbb{Q}$-dynamics.
- **Idea:** *Calibrate* the relevant parameters under $\mathbb{Q}$ (in particular λ) such that the model closely matches the prices of plain vanilla options.
- Use the calibrated parameters to determine arbitrage-free prices of more complicated products.

Recipe for Calibration

- Determine a closed-form solution for option prices that depends on the particular choice of the market price of risk, i.e., an expression

$$C(S_0, \nu_0, \Theta, K, T)$$

for a strike price K, time horizon T, and parameter set $\Theta = (\kappa, \theta, \sigma, \rho, \lambda)$.

- Observe market prices of options $C_1(K_1, T_1), \ldots C_N(K_N, T_N)$ for various combinations of K and T.
- Solve the following minimization problem for a set of weights w:

$$\Theta^* = \arg\min_{\Theta} \sum_{i=1}^{N} w_i \big[C(S_0, \Theta, K_i, T_i) - C_i(K_i, T_i) \big]^2$$

- This shows a potential conflict between estimation and calibration: time series information can be used to determine the parameters κ and σ in the model under $\mathbb{Q}$, and these values might differ from those obtained by calibration.

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Costs and Dividends

- In the theory we assume that assets are self-financing, but, in reality, stocks often generate dividends, and commodities typically bring storage costs.
- Strategy: specify where the dividends go (or where the costs are financed from). In this way, the given asset becomes part of a self-financing portfolio. Then derive the distribution of the asset under a suitable EMM.
- To illustrate, suppose that S_t is the price at time t of a dividend-paying stock, and assume for convenience that dividend is paid continuously at a fixed rate, as a percentage of the stock price. We show two implementations of the strategy above.

- Usual BS model:

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$$
$$\mathrm{d}M_t = rM_t \,\mathrm{d}t$$

but now suppose that the stock pays continuously a fixed-percentage dividend, i.e., the dividend received from one unit of the stock during the instantaneous interval from t to $t + \mathrm{d}t$ is $qS_t\mathrm{d}t$ where q is a constant.

- We can choose to re-invest the dividends into the stock. Let V_t be the value at time t of the portfolio that is created in this way. We have for small Δt:

$$V_{t+\Delta t} = V_t + \frac{V_t}{S_t}\left(S_{t+\Delta t} - S_t\right) + \frac{V_t}{S_t}\, qS_t \Delta t.$$

Dividends in Continuous Time

- In continuous time:

$$\mathrm{d}V_t = \frac{V_t}{S_t}\left(\mathrm{d}S_t + qS_t\,\mathrm{d}t\right) = (\mu + q)V_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t.$$

- The portfolio V_t is self-financing, so under $\mathbb{Q}$:

$$\mathrm{d}V_t = rV_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t^{\mathbb{Q}}.$$

- From $\mathrm{d}V_t = (V_t/S_t)(\mathrm{d}S_t + qS_t\,\mathrm{d}t)$ it follows that $\mathrm{d}S_t = (S_t/V_t)(\mathrm{d}V_t - qV_t\,\mathrm{d}t)$.
- Therefore

$$\mathrm{d}S_t = (r - q)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}.$$

This allows us to price options that are stated in terms of S_t.

- Alternative approach: assume that the dividends are placed in an savings account A.
- We have for small time interval of length Δt:

$$A_{t+\Delta t} = A_t + rA_t\Delta t + qS_t\Delta t$$

so that $\mathrm{d}A_t = (rA_t + qS_t)\,\mathrm{d}t$.

- The portfolio $V_t := S_t + A_t$ is self-financing. So, under $\mathbb{Q}$,

$$\mathrm{d}V_t = rV_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}.$$

- From $\mathrm{d}V_t = \mathrm{d}S_t + \mathrm{d}A_t$ it follows that $\mathrm{d}S_t = \mathrm{d}V_t - \mathrm{d}A_t$.

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- Therefore,

$$
\begin{aligned}
\mathrm{d}S_t &= rV_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}} - (rA_t + qS_t)\,\mathrm{d}t \\
&= r(S_t + A_t)\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}} - (rA_t + qS_t)\,\mathrm{d}t \\
&= (r-q)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^{\mathbb{Q}}.
\end{aligned}
$$

  We find the same SDE for S_t under $\mathbb{Q}$ as was found on the basis of the reinvestment strategy.
- The pricing formula for a call option written on S_t becomes

$$
C_0 = e^{-qT}S_0(d_1) - e^{-rT}K(d_2)
$$

$$
d_1 = \frac{\log(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.
$$

General Setting

- Consider an extension of the generic state space model

$$
\begin{aligned}
\mathrm{d}X_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\
Y_t &= \pi_Y(t, X_t).
\end{aligned}
$$

  by introducing an m-dimensional *dividend process* $D_t = D(t, X_t)$ representing the cumulative dividends of the m assets.
- $\mathrm{d}D_t$ represents the dividends at time t.
- The *gains process* is defined as

$$
G_t = Y_t + D_t
$$

- A process ϕ is called a self-financing trading strategy if

$$
V_t = \phi_t' Y_t, \qquad \begin{aligned} \mathrm{d}V_t &= \phi_t' \mathrm{d}G_t \\ &= \phi_t' \mathrm{d}Y_t + \phi_t' \mathrm{d}D_t \end{aligned}
$$

Discounted Gain Process

- Given a pricing kernel K, we define the deflated price process by $Y^K = KY$.
- What is an appropriate definition for the deflated gains process? $\Longrightarrow$ With dividends, it does not make sense to define the deflated gains process by $G^K = KY + KD$, since it does not take the timing and reinvestment of the dividends into account.
- Instead, we define the deflated gains process G^K s.t. deflated wealth $V^K = KV^\phi$ generated by self-financing trading strategy ϕ equals wealth generated by this trading strategy and deflated prices and gains:

$$V^K = \phi'(KY), \qquad \mathrm{d}V^K = \phi'\mathrm{d}G^K, \qquad G^K \text{ is a } \mathbb{P}\text{-martingale}$$

- We already know $Y^K = KY$. What's about D^K?

Dividend Dynamics

- **Easiest Formulation** (dividends are locally risk-free):

$$\mathrm{d}D_t = \mu_D(t, X_t)\mathrm{d}t$$

Then, the discounted dividends follow (check!):

$$\mathrm{d}D_t^K = K_t \mu_D(t, X_t)\mathrm{d}t$$

- **General Case** (dividends my be driven by systematic or idiosyncratic shocks):

$$\mathrm{d}D_t = \mu_D(t, X_t)\mathrm{d}t + \sigma_D(t, X_t)\mathrm{d}W_t$$

Then, the discounted dividends follow (check!):

$$\mathrm{d}D_t^K = \left[K_t \mu_D(t, X_t) + \sigma_K' \sigma_D\right]\mathrm{d}t + K_t \sigma_D' \mathrm{d}W_t$$

First FTAP with Dividends

- Given: joint process of asset prices $(Y_t)_{t\geq 0}$, cumulative dividends $(D_t)_{t\geq 0}$
- The deflated gains process G^K is given by

$$dG^K = d(KY) + dD_t^K.$$

First Fundamental Theorem of Asset Pricing

The following are equivalent:

1. The market is free of arbitrage.
2. There is a positive adapted scalar process $(K_t)_{t\geq 0}$ such that the process $(G_t^K)_{t\geq 0}$ is a martingale under $\mathbb{P}$.

Pricing with Dividends

- By definition $K_0 = 1$, and $D_0 = 0$.
- FTAP with dividends implies:

$$G_t^K = \mathbb{E}_t[G_T^K] \qquad \iff \qquad Y_t = Y_t^K + D_t^K = \mathbb{E}_t[Y_T K_T + D_T^K],$$

  in particular, $Y_0 = \mathbb{E}[Y_T K_T + D_T^K]$
- **Remark:** The FTAP works for other numéraire-measure-combinations as well. In particular, for $N_t = M_t$:

$$Y_t = \mathbb{E}_t^{\mathbb{Q}}\Big[\frac{Y_T}{M_T} + \int_t^T \frac{1}{M_u}\mathrm{d}D_u\Big]$$

- If dividends follow the dynamics $\mathrm{d}D_t = \mu_D(t, X_t)\,\mathrm{d}t$, then

$$Y_t = \mathbb{E}_t^{\mathbb{Q}}\Big[Y_T\,\mathrm{e}^{-\int_t^T r_s ds} + \int_t^T \mathrm{e}^{-\int_t^u r_s ds}\mu_D(u, X_u)\mathrm{d}u\Big],$$

  i.e., prices have a Feynman-Kac representation.

Part VI

Fixed Income Modeling

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Coupon Bonds

- Now, we are turning to interest rate products beyond a simple money market account.
- Bond = tradeable debt issued by borrower represented by a contract to repay the notional plus interest over the lifetime of the bond.
- Modeling bonds is more involved than modeling stocks because
  1. they pay regular coupons C_i at predefined payment dates $T_i \Rightarrow$ clean vs. dirty prices
  2. they have a finite time horizon T with a known redemption value N
  3. their volatility dies out as $t \to T$
  4. they are exposed to default risk (see Chapter 7) and liquidity risk
- Structure of a coupon bond:

Example: United States of America 1.375% 16/2…

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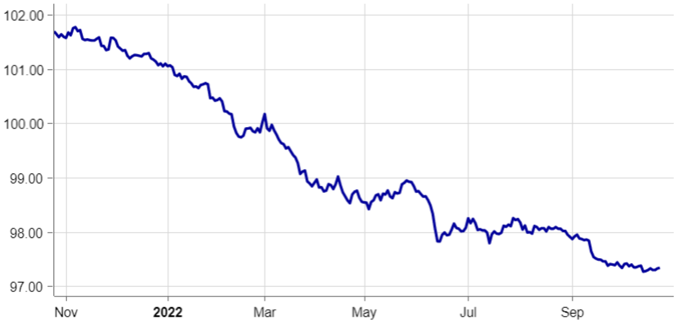

- The graph depicts the evolution of the *clean price*.
- The true market price is the *dirty price* = *clean price* + *accrued interest*.
- Accrued interests are paid to compensate the seller for the period during which the bond has been held but for which she will receive no coupon payment.

Example: United States of America 1.375% 16/2…

- Bond Volatility is dying out as $t \to T$.
- Clean Price $\to N$ as $t \to T$.
- Dirty Price $\to N + C$ as $t \to T$.

Bond Yields

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- The *yield-to-maturity* $y_t^c(T)$ of a coupon bond paying coupons at a rate c ($C = c\Delta_{T_i} N$) and maturing at $T = T_n$ is implicitly defined by

$$P_t^c = \sum_{i=1}^{n} C \mathrm{e}^{-y_t^c(T)(T_i - t)} + N \, \mathrm{e}^{-y_t^c(T)(T-t)}$$

- In practice, bonds are often quoted in terms of yields instead of prices.
- The concept makes the implicit assumption that one can reinvest the coupon payments at the same rate of return.
- Yields of zero-coupon bonds are also called spot rates, i.e., $R_t(T) = y_t^0(T)$.
- Solving for the yield-to-maturity typically requires a computer since closed-form solutions are only available in rare special cases.
- There is an approximation for the *discretely-compounded* yield-to-maturity which admits a nice interpretation:

$$y_{simple} \approx \frac{C}{P_0} + \frac{1}{T-t} \frac{N - P_t}{P_t}$$

Evolution of Bond Yields 1y

source: tradingeconomics.com

Evolution of Bond Yields 10y

source: tradingeconomics.com

Evolution of Bond Yields 50y

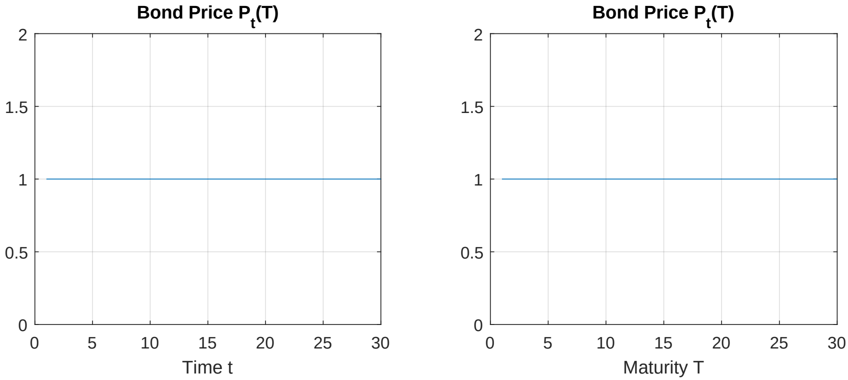
source: tradingeconomics.com

Zero-Coupon Bonds

- A *zero-coupon bond* is a bond that does not pay any coupons.
- A coupon bond is just a portfolio of zero-coupon bonds.
- For our modeling purposes, we consider zero-coupon bonds with notional $N = 1$ only, and assume that these bonds can be traded for every time horizon T. These bonds will be called T-*bonds*.
- The time-t price of a T-bond is denoted by $P_t(T)$. Convention: $P(T) = P_0(T)$.
- This is the discount factor at time t for safe payments made at time T. It represents the “time value of money”.
- Arbitrage-free (dirty) price of a coupon bond that pays coupons C at predefined payment dates T_i, $i = 1, \ldots, n$, has a notional N, and matures at time $T = T_n$:

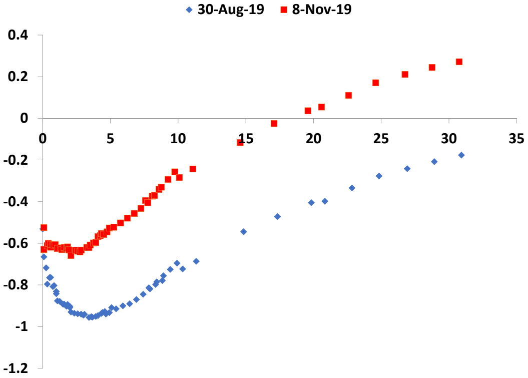
$$P_t^c = \sum_{i=1}^{n} C\, P_t(T_i) + N\, P_t(T)$$

Bond Price versus Discount Curve



Problem: Term Structure of Interest Rates

1. The graph below depicts the yield curve $T \mapsto y_t(T)$ of German Bundesanleihen in 2019. Plot the yield curve of German Bundesanleihen as of 11th of October (data on Canvas).
2. Explain how and why the term structure has been evolving over the last couple of years and why this might be a problem when we model the term structure of interest rates.



Problem: Solution (1)

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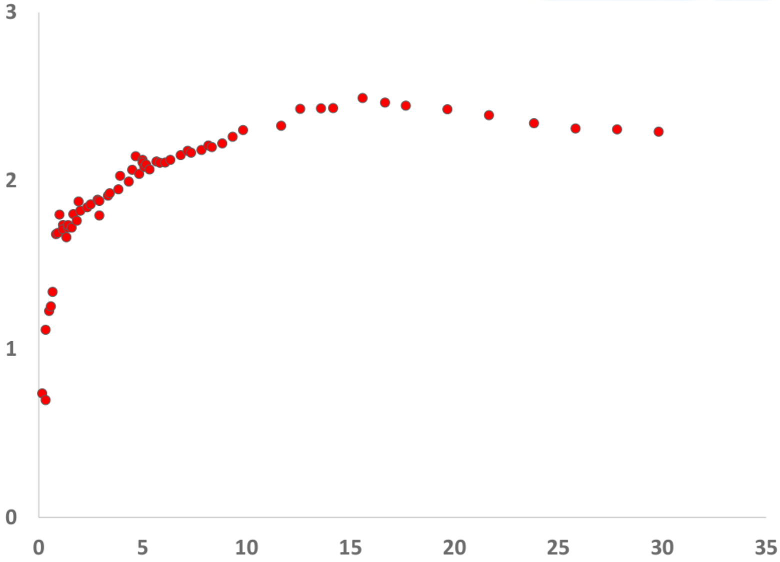

Problem: Solution (2)

Outline for Bond Modeling

We have to deal with five problems:

1. **Term Structure of Interest Rates**
   $\rightarrow$ Model how interest rates vary over time.
2. **Coupon Payments**
   $\rightarrow$ Model the prices of zero-coupon bonds. A coupon bond is just a portfolio of zero-bonds.
3. **Finite Time Horizon**
   $\rightarrow$ We already know how to price derivatives with a finite time horizon.
4. **Vanishing Volatility**
   $\rightarrow$ This problem will be solved automatically.
5. **Credit Risk**
   $\rightarrow$ Add a jump process to the dynamics that models credit default (see Chapter 7).

In order to understand how these steps can be carried out we need to establish the relations between interest rates and bond prices.

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Spot Rates vs. Forward Rates

To make discount factors for different maturities more easily accessible, usually a translation is made to interest rates (or *yields to maturity*). There are two fundamental types of interest rates for each bond issuer.

1. **Spot rate** $R_t(T)$ holds at time t for an investment over $[t, T]$. Convention: $R(T) = R_0(T)$.

2. **Forward rate** $F_t(T_1, T_2)$ holds at time t for an investment over $[T_1, T_2]$. Convention: $F(T_1, T_2) = F_0(T_1, T_2)$.

Spot Rates

- The spot rate can be backed out from zero bonds from the equation

$$P_t(T) = \mathrm{e}^{-R_t(T)(T-t)} \quad \Longleftrightarrow \quad R_t(T) = -\frac{1}{T-t} \ln\left(P_t(T)\right)$$

- Price of a coupon bond that pays coupons C at predefined payment dates T_i, $i = 1, \dots, n$, has a notional N, and matures at time $T = T_n$:

$$P_t^c = \sum_{i=1}^{n} C\mathrm{e}^{-R_t(T_i)(T_i-t)} + N\,\mathrm{e}^{-R_t(T)(T-t)}$$

- The curve that is obtained by plotting $P_t(T)$ against T is called the *discount curve*, i.e., $T \mapsto P_t(T)$
- The curve that is obtained by plotting $R_t(T)$ against T is called the *spot curve*, i.e., $T \mapsto R_t(T)$

Forward Rates and Forward Agreements

- *A forward agreement* is a contract that allows an investor to log in today an interest rate for an investment over a future time interval.
- Forward rate $F_t(T_1, T_2)$ holds at time t for an investment over $[T_1, T_2]$. Convention: $F(T_1, T_2) = F_0(T_1, T_2)$.

- No arbitrage implies

$$\underbrace{e^{R_t(T_1)(T_1-t)}}_{=1/P_t(T_1)} e^{F_t(T_1,T_2)(T_2-T_1)} = \underbrace{e^{R_t(T_2)(T_2-t)}}_{=1/P_t(T_2)}$$

- Consequently,

$$\begin{aligned} F_t(T_1, T_2) &= \frac{1}{T_2 - T_1} \ln\left(\frac{P_t(T_1)}{P_t(T_2)}\right) \\ &= \frac{1}{T_2 - T_1}\left[R_t(T_2)(T_2 - t) - R_t(T_1)(T_1 - t)\right] \end{aligned}$$

Instantaneous Forward Rate

- We define the *instantaneous forward rate* as

$$F_t(T) = \lim_{\Delta t \to 0} F_t(T, T + \Delta t)$$

- An application of L'Hospitals rule yields

$$F_t(T) = -\frac{\partial}{\partial T} \ln P_t(T) = -\frac{P_t'(T)}{P_t(T)}$$

- Since $\ln P_t(T) = -R_t(T)(T - t)$, we obtain

$$F_t(T) = R_t(T) + (T - t)\,\frac{\partial}{\partial T} R_t(T)$$

- The curve that is obtained by plotting $F_t(T)$ against T is called the *forward curve*, i.e., $T \mapsto F_t(T)$

Important Relations

- The discount factors can be expressed in terms of the forward rates

$$P_t(T) = \mathrm{e}^{-\int_t^T F_t(s)ds}$$

- In particular, to ensure that discount factors are monotonically decreasing it is necessary and sufficient that the forward rates are positive.
- We can express the spot rate in terms of the forward rate by

$$R_t(T) = \frac{1}{T-t}\int_t^T F_t(s)\mathrm{d}s$$

- This shows that the spot rates can be viewed as a cumulative average of the forward rates.

- By definition

$$r_t = \lim_{\Delta t \to 0} R_t(t + \Delta t) = - \lim_{\Delta t \to 0} \frac{\partial}{\partial T} \ln P_t(t + \Delta t) = F_t(t)$$

- A zero-bond with maturity at T can be considered as a "derivative" with constant payoff 1 at T, i.e.,

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{M_t}{M_T} \cdot 1\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]$$

- We thus need appropriate models for the short rate. From these, we can derive
  - (Zero)-coupon bond prices
  - Term structure of interest rates, i.e., the mapping $T \to R_t(T)$
  - Prices of interest rate derivatives

LIBOR Rates

- London Interbank Offered Rate (LIBOR) is an interest-rate average calculated from estimates submitted by the leading banks in London.
- The real-world LIBOR rates are simple interest rates without compounding during their lifetime with maturity in 1 day, 1 month, 3 months, 6 months, and 12 months.
- In this lecture, we refer to LIBOR as a set of discretely compounded risk-free rates.

- Tenor: $\Delta_{T_i} = T_{i+1} - T_i$
- current LIBOR-spot rate for $[t, T_i]$: $L_t(t, T_i)$
- current LIBOR-forward rate for $[T_i, T_j]$: $L_t(T_i, T_j)$
- future LIBOR-spot rate for $[T_i, T_j]$: $L_{T_i}(T_i, T_j)$, $T_i > t$

LIBOR Rates

- Under no arbitrage, the LIBOR-forward rates satisfy

$$1 + L_t(T_i, T_j)(T_j - T_i) = e^{F_t(T_i,T_j)(T_j - T_i)} = \frac{P_t(T_i)}{P_t(T_j)}$$

$$\Longrightarrow \qquad L_t(T_i, T_j) = \frac{1}{T_j - T_i}\left[\frac{P_t(T_i)}{P_t(T_j)} - 1\right].$$

- Using $\Delta_{T_i} = T_{i+1} - T_i$, the one-period LIBOR-forward rates satisfy

$$L_t(T_i) = L_t(T_i, T_{i+1}) = \frac{1}{\Delta_{T_i}}\left[\frac{P_t(T_i)}{P_t(T_{i+1})} - 1\right]$$

- LIBOR-spot rates:

$$L_{T_i}(T_i, T_j) = \frac{1}{T_j - T_i}\left[\frac{1}{P_{T_i}(T_j)} - 1\right]$$

and the corresponding one-period rate

$$L_{T_i} = L_{T_i}(T_i, T_{i+1}) = \frac{1}{\Delta_{T_i}}\left[\frac{1}{P_{T_i}(T_{i+1})} - 1\right]$$

.

Floating Rate Notes

- A *Floating Rate Note* is a bond with variable coupon payments that are typically linked to a reference rate.
- It is very common in quantitative finance to use LIBOR rates as reference interest rates.
- Variable coupon payments made at times T_i, $i = 1, \dots, n$ with $\Delta_{T_i} = T_{i+1} - T_i$, are spot LIBOR payments $L_{T_{i-1}} = L_{T_{i-1}}(T_{i-1}, T_i)$ fixed at the *previous* payment date T_{i-1}.

- Payment structure of a FRN:

t	T_1	T_2	$\dots$	T_{n-1}	$T = T_n$
C_t	$L_{T_0}\Delta_{T_0}N$	$L_{T_1}\Delta_{T_1}N$	$\dots$	$L_{T_{n-2}}\Delta_{T_{n-2}}N$	$(1 + L_{T_{n-1}}\Delta_{T_{n-1}})N$

Price of a Floating Rate Note

- Determine the price of the FRN at time T_{n-1}:

$$P^{float}_{T_{n-1}} = P_{T_{n-1}}(T_n)N(1 + L_{T_{n-1}}\Delta_{T_{n-1}}) = \frac{N(1 + L_{T_{n-1}}\Delta_{T_{n-1}})}{1 + L_{T_{n-1}}\Delta_{T_{n-1}}}$$

$$\Longrightarrow \quad P^{float}_{T_{n-1}} = N.$$

- Determine $P^{float}_{T_{n-2}}$ by discounting value components at T_{n-1}
  - value of remaining cash flows: N
  - coupon: $L_{T_{n-2}}N$

  discounting yields

$$P^{float}_{T_{n-2}} = \frac{N(1 + L_{T_{n-2}}\Delta_{T_{n-2}})}{1 + L_{T_{n-2}}\Delta_{T_{n-2}}}$$

$$\Longrightarrow \quad P^{float}_{T_{n-2}} = N.$$

Therefore (mathematical induction): $P^{float}_{T_j} = N$. One can also show $P^{float}_{t} = N$ for all $t \leq T$.

Interest Rate Swap

- An interest rate swap is a derivative contract which exchanges one stream of future interest payments for another stream based on a specified principal amount. Interest rate swaps usually involve the exchange of a fixed interest rate $s(T)$ for a floating rate L_t.

- How should the *par swap rate* $s(T)$ be chosen such that the price of the contract is zero at initiation?
- An interest rate swap is equivalent to the exchange of the coupon payments (but not the notionals) of a coupon bond against those of a floating rate note.

Par Swap Rate

- The swap rate must be chosen such that both products have the same price

$$\underbrace{P_0^s(T)}_{\text{Price of a Coupon bond}} \stackrel{!}{=} \underbrace{N}_{\text{Price of a FRN}}$$

- Choose $s(T)$ such that the market is free of arbitrage, i.e.,

$$N = \sum_{i=1}^{n} s_0(T)\Delta_{T_{i-1}} N P_0(T_i) + N P_0(T)$$

$$\implies \quad 1 = \sum_{i=1}^{n} s_0(T)\Delta_{T_{i-1}} P_0(T_i) + P_0(T)$$

$$\implies \quad s_0(T) = \frac{1 - P_0(T)}{\sum_{i=1}^{n} \Delta_{T_{i-1}} P_0(T_i)}$$

- The mapping $T \mapsto s_t(T)$ is the *swap curve* at time t.

Value of a Swap

- While the par swap rate $s_0(T)$ is chosen such that the value of the swap at initiation is zero, the swap value will be changing over time.
- We denote the time-t value of a payer swap (i.e., holder is the counterparty that pays the fixed interest) by V_t^{payer}. By construction $V_0^{payer} = 0$.
- If $t > 0$, the value of this swap equals the difference between the floating leg and the fixed leg, i.e.,

$$\begin{aligned} V_t^{payer} &= \quad V_t^{float} &&- \quad V_t^{fixed} \\ &= \quad N[1 - P_t(T)] &&- \quad s_0(T) \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i) \\ &= s_t(T) \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i) &&- \quad s_0(T) \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i) \end{aligned}$$

Value of a Swap

- Consequently, the value of a payer swap is

$$V_t^{payer} = [s_t(T) - s_0(T)] \sum_{i=1}^{n} \Delta_{T_{i-1}} N P_t(T_i)$$

- The value of a receiver swap (holder pays variable interest) at time t is just $V_t^{receiver} = -V_t^{payer}$.
- **Moral**: Swaps can be priced without an interest rate model. All we need is the empirically observable discount curve, i.e., prices of zero-coupon bonds.
- A payer *swaption* is a contract that entitles the holder to enter, at a given time in the future, a payer swap with a specified duration and a swap rate that is determined in advance (the strike).
- To price swaptions, we need a model that describes the evolution of the swap curve over time. $\longrightarrow$ Swap Market Model.

Bond Options

- A European bond option is a contract between two counterparties, whereby the buyer (holder) has the right to buy (Call option) or to sell (Put option) the underlying bond from/to the seller (stillholder) at a predetermined strike price K at its maturity T_1.
- Option with maturity in T_1 on a zero bond with maturity in $T_2 > T_1$:

$$
\begin{aligned}
Call_{T_1}(P_{T_1}(T_2)) &= (P_{T_1}(T_2) - K)^+ \\
Put_{T_1}(P_{T_1}(T_2)) &= (K - P_{T_1}(T_2))^+
\end{aligned}
$$

- Put-call-parity for European bond options

$$
Put_t = Call_t - P_t(T_2) + K \cdot P_t(T_1).
$$

- To price bond options, we need a model that describes the evolution of the bond prices over time. $\longrightarrow$ Short Rate models, HJM framework.

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- Interest rate options are options where the underlying is an interest rate.
- If the underlying interest rate exceeds (caplet) or falls below (floorlet) a certain boundary at maturity, the holder of the option can claim an interest payment.
- Caplet with maturity T_i and strike rate L_C on a notional N has payoff at time T_i:

$$(L_{T_{i-1}} - \underbrace{L_C}_{\text{strike}})^+ \Delta_{T_{i-1}} N$$

- Cap: Portfolio of caplets
  $\Rightarrow$ hedge against increasing interest rates
- Floor: Portfolio of floorlets with payoffs $(L_F - L_{T_{i-1}})^+ \Delta_{T_{i-1}} N$
  $\Rightarrow$ hedge against decreasing interest rates.
- To price swaptions, we need a model that describes the evolution of the LIBOR rates over time. $\longrightarrow$ LIBOR Market Model.

Relation between Interest Rate Options and Swaps

- While an interest rate swap provides a perfect hedge against fluctuating interest rates, a caplet only insures against rising interest rates and a floorlet against shrinking interest rates.
- Consider a long-short portfolio of caplets and floorlets with identical strike rates $\overline{L} = L_C = L_F$:

$$
\begin{aligned}
&\left[(L_{T_{i-1}} - \overline{L})^+ - (\overline{L} - L_{T_{i-1}})^+\right]\Delta_{T_{i-1}} N \\
=&\left[\max(L_{T_{i-1}}, \overline{L}) - \overline{L} - \max(L_{T_{i-1}}, \overline{L}) + L_{T_{i-1}}\right]\Delta_{T_{i-1}} N \\
=&\left[L_{T_{i-1}} - \overline{L}\right]\Delta_{T_{i-1}} N \\
=&L_{T_{i-1}}\Delta_{T_{i-1}} N - \overline{L}\Delta_{T_{i-1}} N
\end{aligned}
$$

- This is identical to an exchange of a variable interest rate and a fixed interest rate, i.e., a one-period interest rate swap.
- Interest rate swaps can thus be decomposed into a long-short portfolio of caps and floors. "Cap – Floor = Payer Swap"

- Caplet with maturity T_i and strike rate L_C on a notional N has payoff at time T_i:

$$
\begin{aligned}
&(L_{T_{i-1}} - L_C)^+ \Delta_{T_{i-1}} N \\
=&\left(\frac{1}{\Delta_{T_{i-1}}}\left[\frac{1}{P_{T_{i-1}}(T_i)} - 1\right] - L_C\right)^+ \Delta_{T_{i-1}} N \\
=&\left(\frac{1}{P_{T_{i-1}}(T_i)} - 1 - \Delta_{T_{i-1}} L_C\right)^+ N
\end{aligned}
$$

- The caplet value at the fixing date T_{i-1} is

$$
\left(1 - P_{T_{i-1}} - P_{T_{i-1}}\Delta_{T_{i-1}} L_C\right)^+ N = \left(N - P_{T_{i-1}}(1 + \Delta_{T_{i-1}} L_C)N\right)^+
$$

- A caplet can be viewed as a put option on a zero-coupon bond that matures at time T_i with face value $(1 + \Delta_{T_{i-1}} L_C)N$. The expiry date of the option is T_{i-1}, and the strike is N.

Table of Contents

Modeling the Term Structure of Interest Rates

- We first consider default-free (and perfectly liquid) bonds corresponding to the discount factors and interest rates.
- We start with the benchmark no arbitrage Vasicek model.
- We then generalize this benchmark model, focusing on so-called affine term structure models.
- We will also study the Heath-Jarrow-Morton framework and the LIBOR market model.
- The pricing of bonds can be influenced significantly by credit risk (and liquidity risk) $\longrightarrow$ Chapter 7.

Wishlist for Term Structure Models

A good term-structure model should be able to

- reproduce the currently observed term structure (i.e., bond prices).
- reproduce currently observed prices of other term structure products.
- generate (under $\mathbb{P}$) reasonable future term structures (for instance does not generate (very) negative interest rates).
- capture volatilities of rates for different maturities and correlations between them.
- be tractable; allows quick pricing of popular term structure derivatives such as swaptions and interest rate caps.

- A generic short-rate model for the evolution of the term structure can be written as follows:

$$\mathrm{d}X_t = \mu_X(t, X_t)\mathrm{d}t + \sigma_X(t, X_t)\mathrm{d}W, \qquad r_t = h(t, X_t)$$

- Money Market Account: $\mathrm{d}M_t = M_t r_t \mathrm{d}t$
- A T-bond is just a derivative with constant payoff $P_T(T) = 1$ at maturity T. Pricing under $\mathbb{Q}$:

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{M_t}{M_T} \cdot 1\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]$$

- The TSIR is thus given by

$$R_t(T) = -\frac{1}{T-t} \log \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]$$

Vasicek model under $\mathbb{P}$ and under $\mathbb{Q}$

- Vasicek (1977) originally chosed an Ornstein-Uhlenbeck process for the short rate under $\mathbb{P}$:

$$dr_t = a(b - r_t)\,dt + \sigma\,dW_t, \qquad dM_t = M_t r_t dt$$

- This model ($X = r$, $Y = M$) satisfies the NA criterion and λ can be chosen arbitrarily.
- *Assuming* that the market price of risk associated to W_t is a constant λ yielding $dW_t^{\mathbb{Q}} = \lambda\,dt + dW_t$ (where $W_t^{\mathbb{Q}}$ is a BM under $\mathbb{Q}$), and

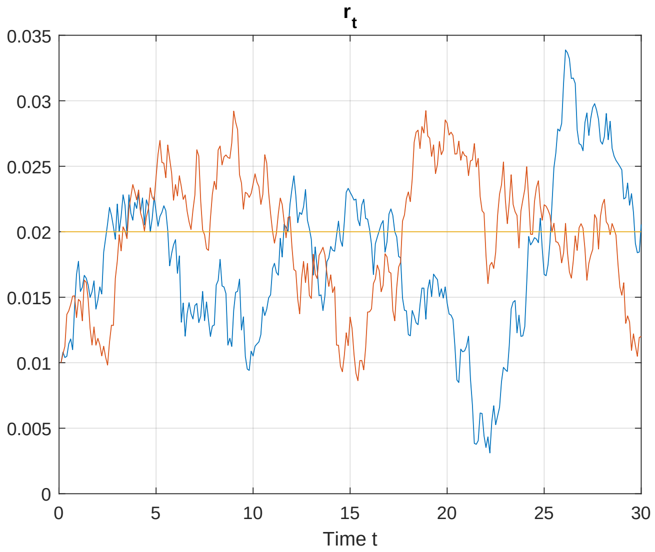
$$dr_t = [a(b - r_t) - \sigma\lambda]\,dt + \sigma\,dW_t^{\mathbb{Q}}$$

which can be written in the form

$$dr_t = a(b^{\mathbb{Q}} - r_t)\,dt + \sigma\,dW_t^{\mathbb{Q}}, \qquad b^{\mathbb{Q}} = b - \lambda\,\frac{\sigma}{a}\,.$$

- This is the model under $\mathbb{Q}$ as we used it before.

"Typical" Paths of the Vasicek Model



Problem: Solving Ornstein-Uhlenbeck

- Show the following properties of the Ornstein-Uhlenbeck process $\mathrm{d}X_t = a(b - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$:

  1. $X_t = X_0 \mathrm{e}^{-at} + b(1 - \mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-a(t-s)} \mathrm{d}W_s$
  2. $X_t \sim \mathcal{N}\left(\mu(X_t), \sigma(X_t)^2\right)$ with

     $\mu(X_t) = X_0 \mathrm{e}^{-at} + b(1 - \mathrm{e}^{-at})$ and $\sigma^2(X_t) = \frac{1-\mathrm{e}^{-2at}}{2a}\sigma^2$

- **Solution**:

Problem: Solving Ornstein-Uhlenbeck

Problem: Solving Ornstein-Uhlenbeck

Bond Price in the Vasicek Model

- We know that the Vasicek model is free of arbitrage, hence we can formulate it under $\mathbb{Q}$:

$$\mathrm{d}r_t = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}}, \qquad b^{\mathbb{Q}} = b - \lambda\,\frac{\sigma}{a}\,.$$

- We know that the price of a T-bond is just a derivative with constant payoff $P_T(T) = 1$ at maturity T. Pricing under $\mathbb{Q}$:

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{M_t}{M_T}\cdot 1\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^T r_s ds}\right]$$

- Question: What would be the pricing relation under $\mathbb{P}$?

- We first calculate $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{M_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right]$.

Bond Price in the Vasicek Model

- Short rate dynamics: $\mathrm{d}r_t = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}}$
- Dynamics of the log-MMA: $\mathrm{d}\log M_t = r_t \mathrm{d}t$
- Consequently,

$$\mathrm{d}(r_t + a\log M_t) = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}} + a r_t \mathrm{d}t = a b^{\mathbb{Q}}\,\mathrm{d}t + \sigma\, dW_t^{\mathbb{Q}}$$

- Integrating and some algebra yields:

$$\log M_t = \frac{1}{a}\Big[a b^{\mathbb{Q}} t + \sigma W_t^{\mathbb{Q}} - (r_t - r_0)\Big]$$

- We know that $r_t = r_0 \mathrm{e}^{-at} + b^{\mathbb{Q}}(1 - \mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-a(t-s)} \mathrm{d}W_s^{\mathbb{Q}}$. Substituting this solution into $\log M_t$ yields

$$\log M_t = \frac{1}{a}\Big[a b^{\mathbb{Q}} t + \sigma W_t^{\mathbb{Q}} + r_0 - \Big(r_0 \mathrm{e}^{-at} + b^{\mathbb{Q}}(1 - \mathrm{e}^{-at}) + \sigma \int_0^t \mathrm{e}^{-a(t-s)} \mathrm{d}W_s^{\mathbb{Q}}\Big)\Big]$$

Bond Price in the Vasicek Model

- Therefore, $\log M_t$ follows a normal distribution under $\mathbb{Q}$ with

$$\mathbb{E}^{\mathbb{Q}}[\log M_t] = b^{\mathbb{Q}} t + \frac{1}{a}\left(1 - \mathrm{e}^{-at}\right)\left(r_0 - b^{\mathbb{Q}}\right)$$

$$\begin{aligned} \operatorname{var}^{\mathbb{Q}}[\log M_t] &= \frac{\sigma^2}{a^2} \int_0^t \left[1 - \mathrm{e}^{-a(t-s)}\right]^2 \mathrm{d}s \\ &= \frac{\sigma^2}{a^2}\left[t - \frac{2}{a}\left(1 - \mathrm{e}^{-at}\right) + \frac{1}{2a}\left(1 - \mathrm{e}^{-2at}\right)\right] \end{aligned}$$

- In turn, $-\log M_T$ is normally distributed as well.
- Now, we can calculate $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{M_T}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right]$, where $\mathrm{e}^{-\log M_T}$ is log-normally distributed, i.e.,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right] = \mathrm{e}^{-\mathbb{E}^{\mathbb{Q}}[\log M_T] + \frac{1}{2} var^{\mathbb{Q}}[\log M_T]}$$

- Substituting everything we know into this expression, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\log M_T}\right] = \exp\Big(-\Big[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\Big]T - \frac{1-\mathrm{e}^{-aT}}{a}\Big[r_0 - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\Big]\Big) \cdot \exp\Big(\frac{\sigma^2}{2a^2}\frac{1-\mathrm{e}^{-2aT}}{2a}\Big)$$

- In turn, the current price of a T-bond in the Vasicek model is

$$P_0(T) = \exp\Big(-\Big[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\Big]T - \frac{1-\mathrm{e}^{-aT}}{a}\Big[r_0 - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\Big]\Big) \cdot \exp\Big(\frac{\sigma^2}{2a^2}\frac{1-\mathrm{e}^{-2aT}}{2a}\Big)$$

with $b^{\mathbb{Q}} = b - \frac{\sigma\lambda}{a}$.

- The yield curve now follows straightforwardly:

$$
\begin{aligned}
R_0(T) &= -\frac{1}{T}\log P_0(T) \\
&= \left[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\right] + \frac{1-\mathrm{e}^{-aT}}{aT}\left[r_0 - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\right] - \frac{\sigma^2}{2a^2}\frac{1-\mathrm{e}^{-2aT}}{2aT}
\end{aligned}
$$

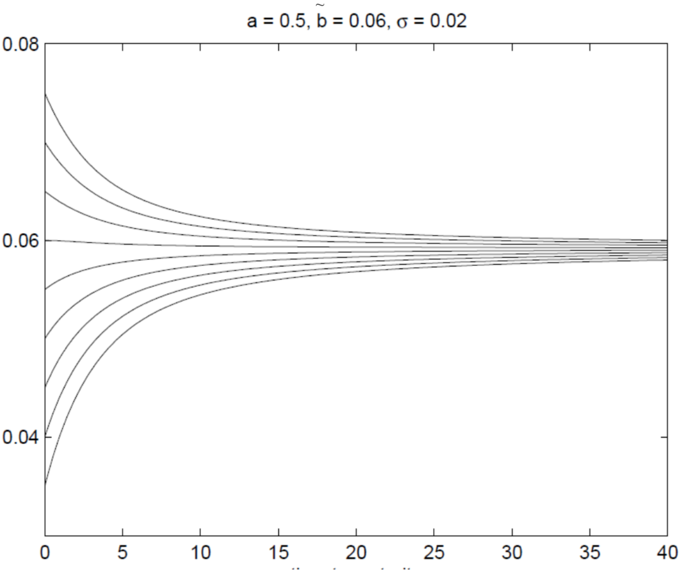
- Taking the limit for super long-term bonds, i.e., $T \to \infty$

$$
\overline{R}_0 := \lim_{T\to\infty} R_0(T) = b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}
$$

- Therefore,

$$
R_0(T) = \overline{R}_0 + \frac{1-\mathrm{e}^{-aT}}{aT}(r_0 - \overline{R}_0) + \frac{\sigma^2}{2a^2}\frac{(1-\mathrm{e}^{-aT})^2}{2aT}
$$

The Yield Curve



Empirical Performance

Tilburg University School of Economics and Management

- Due to its normality property the Vasicek model is very tractable both analytically and numerically. In particular, the model can be simulated exactly by the Euler-scheme.
- The empirical performance of the Vasicek model is bad.
  - The current, observed term structure typically is not matched very well, i.e.,

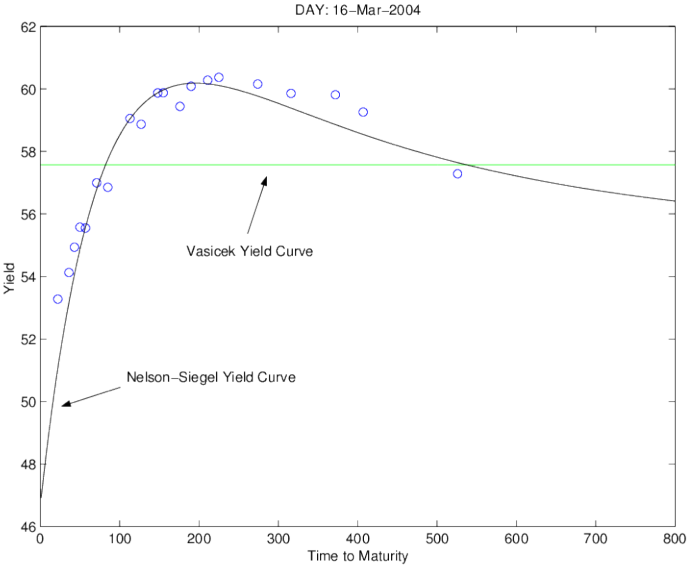
$$R_0(T) = \overline{R}_0 + \frac{1 - \mathrm{e}^{-aT}}{aT}(r_0 - \overline{R}_0) + \frac{\sigma^2}{2a^2}\frac{(1 - \mathrm{e}^{-aT})^2}{2aT}$$

    is typically not very close to the observed one at time $t = 0$
  - This is particularly pronounced if the term structure has a hump.
- This issue can be addressed by the Hull-White model

$$\mathrm{d}r_t = a(t)\big(b^{\mathbb{Q}}(t) - r_t\big)\,\mathrm{d}t + \sigma(t)\,dW_t^{\mathbb{Q}}$$

Using this approach we can "fit the initial term structure".

Empirical Performance



Remark: (Non)-negativity

- In the Vasicek model, interest rates (yields) can become negative without lower bound.
- This issue can be addressed by the Cox-Ingersol-Ross model

$$\mathrm{d}r_t = a\big(b^{\mathbb{Q}} - r_t\big)\,\mathrm{d}t + \sigma\sqrt{r_t}\,dW_t^{\mathbb{Q}},$$

  which ensures that interest rates stay positive.
- One might want to have negative interest rates, but with a lower bound, e.g.,

$$\mathrm{d}X_t = a\big(b^{\mathbb{Q}} - X_t\big)\,\mathrm{d}t + \sigma\sqrt{X_t}\,dW_t^{\mathbb{Q}}, \qquad r_t = X_t - \ell$$

- The CIR model is much less tractable than the Vasicek model (calculations get much more involved, SDE does not possess an explicit solution, distribution is non-central χ^2, and simulation is challenging).

Remarks

- We have only studied the case $t = 0$, but this procedure also works for $t > 0$.
- We obtain

$$P_t(T) = \exp\left(-\left[b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\right](T-t) - \frac{1 - \mathrm{e}^{-a(T-t)}}{a}\left[r_t - b^{\mathbb{Q}} + \frac{\sigma^2}{a^2}\right]\right) \cdot \exp\left(\frac{\sigma^2}{2a^2}\frac{1 - \mathrm{e}^{-2a(T-t)}}{2a}\right)$$

- Consequently, the price can be written as

$$P(t, r; T) = \exp\left(A(t, T) + B(t, T) r_t\right)$$

  for functions $A(t, T)$ and $B(t, T) = -\frac{1}{a}\left(1 - \mathrm{e}^{-a(T-t)}\right)$.
- Any short rate model that leads to such a representation of the bond prices will be called an affine short rate model.

Problem: Estimation of the Vasicek Model

- A standard way to estimate the process r_t under $\mathbb{P}$ is to run a regression

$$r_{t+\Delta t} = \alpha + \beta r_t + \varepsilon_{t+\Delta t},$$

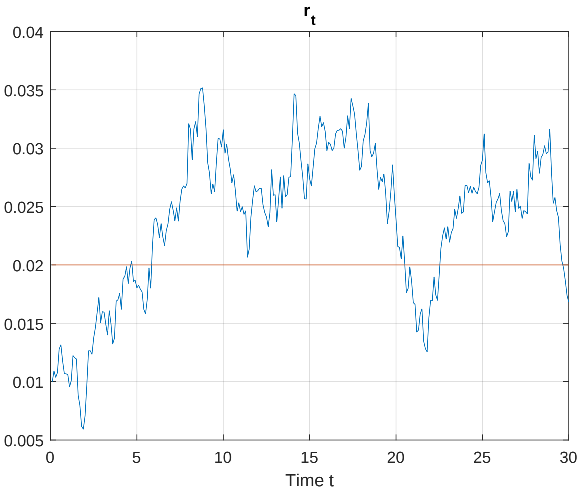
estimated using OLS (under the usual assumptions).

  1. What is the link between α, β, and $s^2 = \text{var}(\varepsilon_{t+\Delta t})$ and a, b, and σ?
  2. Implement a code that estimates the parameters a, b, and σ for given interest rate data and visualize the regression.
  3. Simulate trajectories for the Vasicek model estimated in (2).

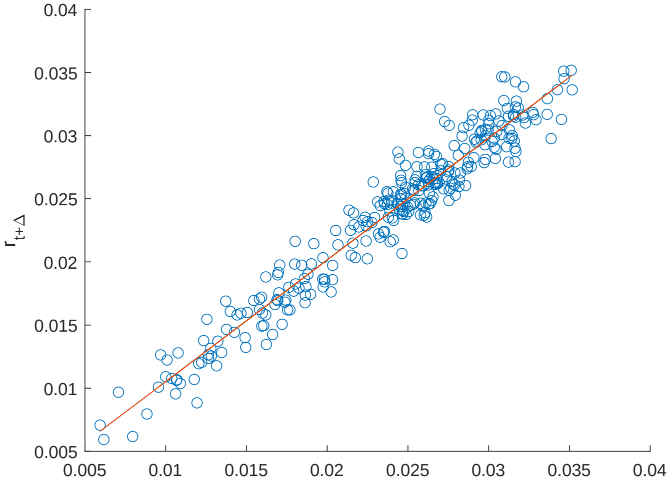
- **Solution:** (1)

Problem: Estimation of the Vasicek Model

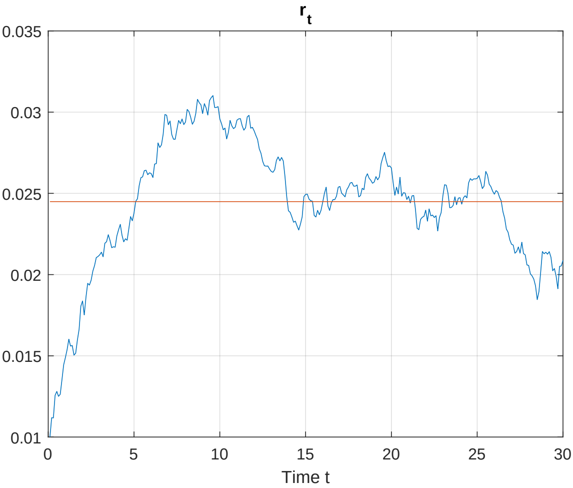
Problem: Data



(Simulated) short rate data generated with $r_0 = 0.01$, $a = 0.25$, $b = 0.02$, $\sigma = 0.015$, $\Delta t = 0.1$.



Simulated model with $\widehat{\alpha} = 8.6844e - 04$, $\widehat{\beta} = 0.9645$, $\widehat{s} = 0.015$



Regression line with $\widehat{a} = -\frac{\log \widehat{\beta}}{\Delta t} = 0.3610$, $\widehat{b} = \frac{\widehat{\alpha}}{1-\widehat{\beta}} = 0.0245$,

$\sigma = \widehat{s}\sqrt{2\widehat{a}/(1 - e^{-2\widehat{a}\Delta t})} = 0.005$.

Recovering observed bond prices

- One obvious drawback of the Vasicek model is that it in general does not match observed bond prices. We describe a way to mend this which actually can be applied to *any* term structure model.
- Consider a term structure model of the general form

$$\begin{aligned} dX_t &= \mu_X(t, X_t)\,\mathrm{d}t + \sigma_X(t, X_t)\,\mathrm{d}W_t \\ r_t &= h(t, X_t). \end{aligned}$$

- Suppose that the forward curve at current time 0 as produced by the model ($F_0^{un}(T)$; "un" for "unadjusted") does not match the observed forward curve ($F_0^{obs}(T)$). Modify the model as follows:

$$\begin{aligned} dX_t &= \mu_X(t, X_t)\,dt + \sigma_X(t, X_t)\,dW_t \\ r_t &= h(t, X_t) + F_0^{obs}(t) - F_0^{un}(t). \end{aligned}$$

- Now the model does match the observed forward curve, and hence also the spot yield curve.

Simplest example

- The simplest term structure model is the one in which the short rate is constant: $r_t = r$. The forward curve is given in this case by

$$F_0^{un}(T) = -\frac{\mathrm{d}}{\mathrm{d}T} \log P_0(T) = -\frac{\mathrm{d}}{\mathrm{d}T} \log e^{-rT} = r.$$

- Using the recipe described on the previous slide, we can modify the model so that it matches the current term structure. The modified short rate model is:

$$r_t = F_0^{obs}(t).$$

- This is still a deterministic model. It matches currently observed bond prices. But it will not match the prices of swaptions, for instance.

Adjusting the Vasicek model

- Now take the Vasicek model (under $\mathbb{Q}$)

$$\mathrm{d}r_t = a(b^{\mathbb{Q}} - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}.$$

- The corresponding forward curve at time 0 is

$$F_0^{un}(r_0, T) = e^{-aT} r_0 + \left(1 - e^{-aT}\right) b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2.$$

- The modified version that matches the current term structure is (rename the original r_t to X_t)

$$dX_t = a(b^{\mathbb{Q}} - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}$$

$$r_t = X_t + F_0^{obs}(t) - F_0^{un}(X_0, t).$$

- The modified Vasicek model can be rewritten by taking the differential of r_t:

$$
\begin{aligned}
\mathrm{d}r_t &= \mathrm{d}X_t + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{obs}(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)\,\mathrm{d}t \\
&= a(b^{\mathbb{Q}} - X_t)\,\mathrm{d}t + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{obs}(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}} \\
&= a\big(b^{\mathbb{Q}} - r_t + F_0^{obs}(t) - F_0^{un}(t)\big)\,\mathrm{d}t \\
&\qquad + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{obs}(t)\,\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}.
\end{aligned}
$$

- To compute $aF_0^{un}(t) + \frac{\mathrm{d}}{\mathrm{d}t}F_0^{un}(t)$, use:

$$
\left(a + \frac{\mathrm{d}}{\mathrm{d}t}\right)(e^{-at}) = 0.
$$

Result: Hull-White model

- From $F_0^{un}(T) = e^{-aT} r_0 + \left(1 - e^{-aT}\right) b^{\mathbb{Q}} - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2$ we get

$$aF_0^{un}(t) + \frac{\mathrm{d}}{\mathrm{d}t} F_0^{un}(t) = ab^{\mathbb{Q}} - \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

- The modified Vasicek model becomes

$$\mathrm{d}r_t = \left(\theta(t) - ar_t\right)\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}$$

with

$$\theta(t) = aF_0^{obs}(t) + \frac{\mathrm{d}}{\mathrm{d}t} F_0^{obs}(t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

- This is known as the *one-factor Hull-White model*.

Affine Term Structure Models

- A term structure model is said to be affine if the yield curves that it produces are of the form

$$R_t(T) = \alpha(t, T) + \beta(t, T)' X_t$$

or equivalently,

$$P_t(T) = e^{A(t,T) + B(t,T)' X_t}$$

with $\alpha(t, T) = -\frac{A(t,T)}{T-t}$, $\beta(t, T) = -\frac{B(t,T)}{T-t}$

- Notation:

$$\begin{aligned} \alpha(t, T) &: \text{ scalar} \\ \beta(t, T) &: \text{ vector of length } n \\ X_t &: n\text{-dimensional process of state variables at time } t \end{aligned}$$

Examples for Affine Term Structure Models

- A sufficient condition for a model to be affine is

$$dX_t = \left(\tilde{A}(t)X_t - g(t)\right)dt + \tilde{B}(X_t)dW_t^{\mathbb{Q}}, \qquad r_t = h(t)'X_t$$

- Notation:

$$\begin{aligned} X_t &: n\text{-dimensional process of state variables at time } t \\ \tilde{A}(t) &: n \times n\text{-matrix} \\ \tilde{B}(X_t) &: n \times k \text{ matrix such that } \tilde{B}(X_t)\tilde{B}(X_t)' \text{ is affine in } X_t \\ g(t), h(t) &: \text{ vectors of length } n \\ W^{\mathbb{Q}} &: k\text{-dimensional standard Brownian motion under } \mathbb{Q} \end{aligned}$$

- Examples ($r_t = X_t$):
  - Black-Karasinski: $d(\log X_t) = a(b_t^{\mathbb{Q}} - \log X_t)\, dt + \sigma\, dW_t^{\mathbb{Q}}$
  - CIR: $dX_t = a(b^{\mathbb{Q}} - X_t)dt + \sigma\sqrt{X_t}dW_t^{\mathbb{Q}}$
  - Dothan: $dX_t = X_t(a^{\mathbb{Q}}dt + \sigma dW_t^{\mathbb{Q}})$
  - Ho-Lee: $dX_t = \sigma^2 t dt + \sigma dW_t^{\mathbb{Q}}$
  - Vasicek / Hull White: $dX_t = a(b^{\mathbb{Q}} - X_t)dt + \sigma dW_t^{\mathbb{Q}}$

- Remember that bond prices are contingent claims on the short rate with terminal value of 1.
- Let $p(t, X; T)$ denote the time-t price of a T-bond. It follows from the Feynman Kac Theorem that bond prices satisfy the following PDE

$$\frac{\partial p}{\partial t} + \nabla p \cdot (\tilde{A}X - g) + \tfrac{1}{2}\mathrm{tr}\Big(H_p\, \tilde{B}(X)\tilde{B}(X)'\Big) = (h'X)p$$

s.t. $p(T, X; T) = 1$

- Since the model is affine, we can rewrite $\tilde{B}(X)\tilde{B}(X)' = \tilde{C} + \tilde{D}\,X$.

$$\frac{\partial p}{\partial t} + \nabla p \cdot (\tilde{A}X - g) + \tfrac{1}{2}\mathrm{tr}\Big(H_p(\tilde{C} + \tilde{D}\,X)\Big) = (h'X)p$$

- In affine models, bond prices are given by

$$p(t, X; T) = \mathrm{e}^{A(t,T) + B(t,T)'X_t}$$

that can be substituted into the TSE yielding ODEs for A and B s.t. $A(T, T) = B(T, T) = 0$.

Example: Vasicek Revisited

- The TSE is given by

$$\frac{\partial p(t,r;T)}{\partial t} + \frac{\partial p(t,r;T)}{\partial r} a(b^{\mathbb{Q}} - r) + \tfrac{1}{2} \frac{\partial^2 p(t,r;T)}{\partial r^2} \sigma^2 = p(t,r;T) r$$

- Substituting the conjecture into the TSE

$$p[\dot{A}(t,T) + \dot{B}(t,T) r] + pB(t,T) a(b^{\mathbb{Q}} - r) + \tfrac{1}{2} pB(t,T)^2 \sigma^2 = pr$$

- Dividing by p and separating yields

$$\dot{A}(t,T) + B(t,T) ab^{\mathbb{Q}} + \tfrac{1}{2} B(t,T)^2 \sigma^2 + r[\dot{B}(t,T) - aB(t,T) - 1] = 0$$

- We obtain two ODEs s.t. $A(T,T) = B(T,T) = 0$:

$$\begin{aligned} \dot{A}(t,T) + B(t,T) ab^{\mathbb{Q}} + \tfrac{1}{2} B(t,T)^2 \sigma^2 &= 0 \\ \dot{B}(t,T) - aB(t,T) - 1 &= 0 \end{aligned}$$

Example: Vasicek Revisited

- Linear ODE for B: $\dot{B}(t,T) - aB(t,T) - 1 = 0$ (e.g., Feynman-Kac):

$$B(t,T) = \int_t^T \mathrm{e}^{-a(s-t)}(-1)\mathrm{d}s = -\frac{1}{a}\left(1 - \mathrm{e}^{-a(T-t)}\right)$$

- Integrating A:

$$\begin{aligned} A(t,T) &= \int_t^T B(s,T)ab^{\mathbb{Q}} + \tfrac{1}{2}B(s,T)^2\sigma^2\mathrm{d}s \\ &= \dots \end{aligned}$$

- Bond price as it was before

$$P(t,r;T) = \exp\left(A(t,T) + B(t,T)r_t\right).$$

Example: Cox-Ingersol-Ross

- The TSE is given by

$$\frac{\partial p(t,r;T)}{\partial t} + \frac{\partial p(t,r;T)}{\partial r} a(b^{\mathbb{Q}} - r) + \frac{1}{2} \frac{\partial^2 p(t,r;T)}{\partial r^2} \sigma^2 r = p(t,r;T) r$$

- Substituting the conjecture into the TSE

$$p[\dot{A}(t,T) + \dot{B}(t,T) r] + pB(t,T) a(b^{\mathbb{Q}} - r) + \frac{1}{2} pB(t,T)^2 \sigma^2 r = pr$$

- Dividing by p and separating yields

$$\dot{A}(t,T) + B(t,T) ab^{\mathbb{Q}} + r[\dot{B}(t,T) - aB(t,T) + \frac{1}{2} B(t,T)^2 \sigma^2 - 1] = 0$$

- We obtain two ODEs s.t. $A(T,T) = B(T,T) = 0$:

$$\dot{A}(t,T) + B(t,T) ab^{\mathbb{Q}} = 0$$
$$\dot{B}(t,T) - aB(t,T) + \frac{1}{2} B(t,T)^2 \sigma^2 - 1 = 0$$

Example: Cox-Ingersol-Ross

- Now, the ODE for B is much more involved, a so-called Riccati equation.

$$\dot{B}(t,T) - aB(t,T) + \frac{1}{2}B(t,T)^2\sigma^2 - 1 = 0$$

- For constant coefficients, by guessing $B(t,T) = k\frac{\Psi_t}{\Psi}$ for a constant k, and a function Ψ, it can be transformed into a linear second-order ODE with well-known solution.
- In the end, we obtain:

$$B(t,T) = -\frac{2(\mathrm{e}^{\gamma(T-t)} - 1)}{\mathrm{e}^{\gamma(T-t)}(\gamma + a) + \gamma - a}, \qquad \gamma = \sqrt{a^2 + 2\sigma^2}$$

- Integrating A:

$$A(t,T) = \int_t^T B(s,T)ab^{\mathbb{Q}}\mathrm{d}s = \frac{2ab^{\mathbb{Q}}}{\sigma^2}\log\left(\frac{2\gamma\mathrm{e}^{0.5(a+\gamma)(T-t)}}{(\gamma+a)(\mathrm{e}^{\gamma(T-t)} - 1) + 2\gamma}\right)$$

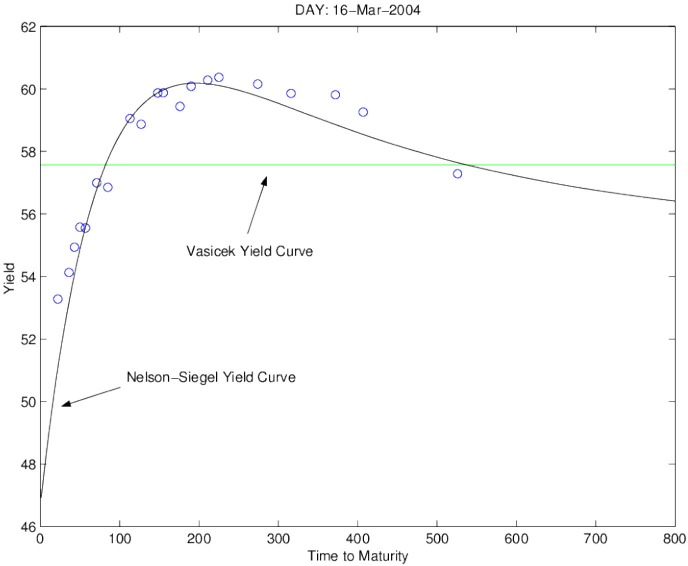
- A Hull-White-type extension of the CIR model would make the calculations extremely messy.

Remarks on Option Pricing

- It is also possible to derive closed-form solutions for European call and put options on zero bonds in affine term structure models.
- The option pricing formulas are very similar to the Black-Scholes formula, but we need another EMM to derive them.
- As for bond prices, the option pricing formula for the CIR is significantly more involved than for the Gaussian models.
- We will address this issue in Section 16.
- It is also possible to derive closed-form option prices for claims on the short rate, i.e., options of the form

$$C(T, r_T) = \Phi(r_T).$$

Empirical Performance



- Single-factor short rate models are not sufficient to model the whole TSIR.
- In the Nelson-Siegel model, the term structure is fitted by a deterministic function with four parameters rather than a dynamic short rate.

$$R_t(T) = \beta_{0,t} + \frac{1 - e^{-a_t(T-t)}}{a_t(T-t)}\beta_{1,t} + \left(\frac{1 - e^{-a_t(T-t)}}{a_t(T-t)} - e^{-a_t(T-t)}\right)\beta_{2,t}$$

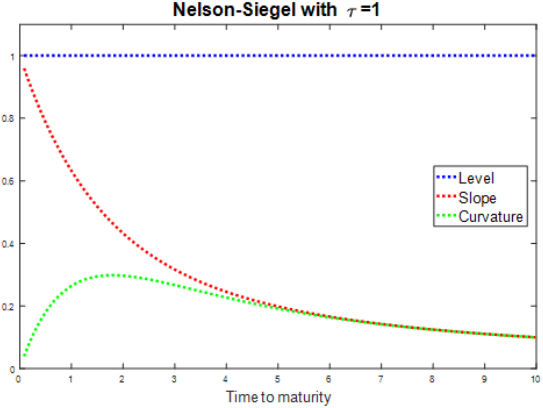
- This implies the forward rate

$$F_t(T) = \beta_{0,t} + e^{-a_t(T-t)}\beta_{1,t} + a_t(T-t)e^{-a_t(T-t)}\beta_{2,t}$$

- We use the notation $\tau_t = 1/a_t$.

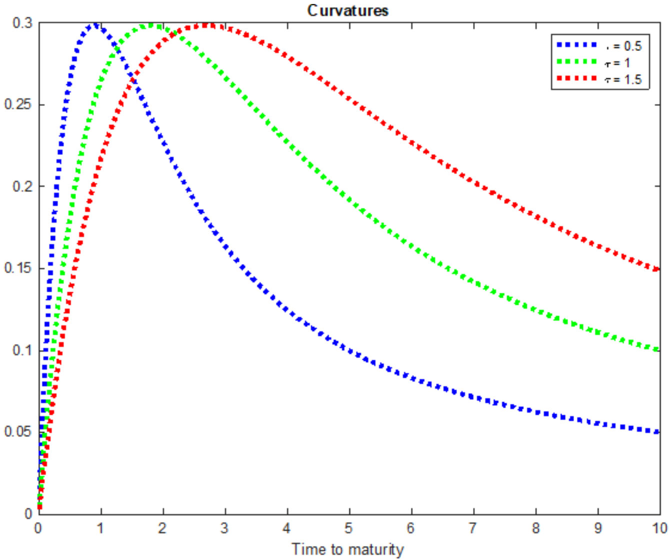
Nelson-Siegel Model

$$R_t(T) = \beta_{0,t} 1 + \frac{1 - e^{-a_t(T-t)}}{a_t(T-t)} \beta_{1,t} + \left(\frac{1 - e^{-a_t(T-t)}}{a_t(T-t)} - e^{-a_t(T-t)} \right) \beta_{2,t}$$



$\beta_{0,t}$: long rate, $\beta_{0,t} + \beta_{1,t}$: short rate, $\beta_{2,t}$: size of hump,
$\tau_t = 1/a_t$: determines the time of hump

Nelson-Siegel Model



Dynamic Nelson-Siegel

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- **Huge drawback**: The Nelson-Siegel term structure cannot be implied by *any* arbitrage-free short-term model.
- **Idea**: Construct a version of the Nelson-Siegel model with factors $\beta_{0,t}, \beta_{1,t}, \beta_{2,t}$ that evolve dynamically over time such that the model reproduces the Nelson-Siegel term structure *as close as possible*.
- Introduce a three-dimensional state process $X_t = (\beta_{0,t}, \beta_{1,t}, \beta_{2,t})'$, and assume

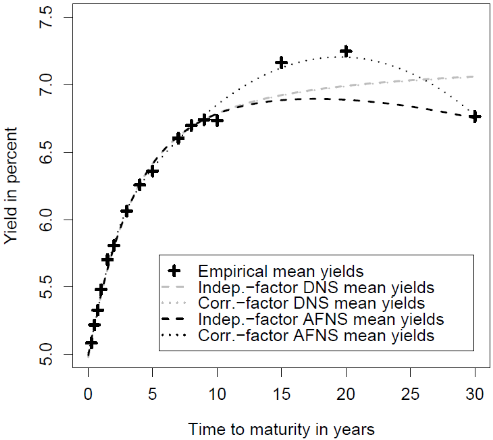
$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}}, \qquad r_t = \rho_0(t) + \rho_1(t)' X_t$$

- One can show that for a particular affine parameter choice (see Christensen et al. 2010), the resulting yield curve is

$$R_t(T) = \beta_{0,t} + \frac{1 - \mathrm{e}^{-a(T-t)}}{a(T-t)}\beta_{1,t} + \Big(\frac{1 - \mathrm{e}^{-a(T-t)}}{a(T-t)} - \mathrm{e}^{-a(T-t)}\Big)\beta_{2,t} - \frac{C(t,T)}{T-t}$$

Dynamic Nelson-Siegel Model

- The resulting model is free of arbitrage, and, due to its affine structure, it has a closed-form solution.
- The empirical performance of this arbitrage-free Nelson-Siegel model (AFNS) is very good.



(Nelson-Siegel-)Svensson Model

- Modification of the Nelson-Siegel Model with six parameters

$$R_t(T) = \beta_{0,t} + \frac{1-\mathrm{e}^{-a_t(T-t)}}{a_t(T-t)}\beta_{1,t} + \left(\frac{1-\mathrm{e}^{-a_t(T-t)}}{a_t(T-t)} - \mathrm{e}^{-a_t(T-t)}\right)\beta_{2,t} + \left(\frac{1-\mathrm{e}^{-b_t(T-t)}}{b_t(T-t)} - \mathrm{e}^{-b_t(T-t)}\right)\beta_{3,t}$$

- This implies the forward rate

$$F_t(T) = \beta_{0,t} + \mathrm{e}^{-a_t(T-t)}\beta_{1,t} + a_t(T-t)\mathrm{e}^{-a_t(T-t)}\beta_{2,t} + b_t(T-t)\mathrm{e}^{-b_t(T-t)}\beta_{3,t}$$

Dynamic (Nelson-Siegel-)Svensson Model

- Like Nelson-Siegel, also Svensson can be turned into a multi-factor model, with four factors.
- The resulting dynamic Svensson model is also not arbitrage-free (by construction) for *any* short-rate model.
- But, the dynamic *four*-factor Svensson model can also be turned into an arbitrage-free affine *five-factor* term structure model. However, it turns out that this requires the introduction of an extra (slope) factor, together with a non-random correction term.

$$
\begin{aligned}
R_t(T) = \beta_{0,t} &+ \frac{1-\mathrm{e}^{-a(T-t)}}{a(T-t)}\beta_{1,t} + \left(\frac{1-\mathrm{e}^{-a(T-t)}}{a(T-t)} - \mathrm{e}^{-a(T-t)}\right)\beta_{2,t} \\
&+ \frac{1-\mathrm{e}^{-b(T-t)}}{b(T-t)}\beta_{4,t} + \left(\frac{1-\mathrm{e}^{-b(T-t)}}{b(T-t)} - \mathrm{e}^{-b(T-t)}\right)\beta_{3,t} \\
&- \frac{C(t,T)}{T-t}
\end{aligned}
$$

Calibration of the Svensson Model

- Given a set of observed bond prices $P_0^{obs}(C, N, T_1, \ldots, T_n)$ at time 0.
- Calibrate the six parameters $\pi = \{\beta_0, \beta_1, \beta_2, \beta_3, a, b\}$ such that theoretical prices

$$P_0^{model}(C, N, T_1, \ldots, T_n) = \sum_{i=1}^{n} C \mathrm{e}^{-R_0(T_i) T_i} + N \mathrm{e}^{-R_0(T_n) T_n}$$

with

$$R_0(T) = \beta_0 + \frac{1 - \mathrm{e}^{-aT}}{aT} \beta_1 + \left(\frac{1 - \mathrm{e}^{-aT}}{aT} - \mathrm{e}^{-aT} \right) \beta_2 + \left(\frac{1 - \mathrm{e}^{-bT}}{bT} - \mathrm{e}^{-bT} \right) \beta_3$$

closely match the observed prices.

Calibration of the Svensson Model

- This can be achieved by an OLS minimization over the parameter set $\pi = \{\beta_0, \beta_1, \beta_2, \beta_3, a, b\}$:

$$\widehat{\pi} = \arg\min_{\pi} \sum_{j=1}^{J} w_j \big[P_0^{obs,j}(C^j, N^j, T_1^j, \ldots, T_n^j) - P_0^{model,j}(C^j, N^j, T_1^j, \ldots, T_n^j) \big]^2$$

- ECB estimates the six Svensson parameters daily.
- The dynamic versions of those models can be estimated by principal component analysis.

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Motivation

- So far, we have studied interest rate models where the short rate r is the only explanatory variable.
- Main advantages:
  - Specifying r as the solution of an SDE allows us to use Markov process theory, so we may work within a PDE framework.
  - In particular it is often possible to obtain analytical formulas for bond prices and derivatives.
- Main disadvantages:
  - It is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short rate model.
  - As the short rate model becomes more realistic, the inversion of the yield curve becomes increasingly more difficult.
- Arbitrage-free Nelson-Siegel Models require more state variables. The HJM-framework goes beyond that idea and models the *whole* forward curve.

Heath-Jarrow-Morton

- The HJM-framework is not a specific model, but a framework for modeling the forward rates.
- We will see that the framework contains the short-rate models as special cases.
- $\mathbb{P}$-dynamics of the forward curve:

$$dX_t = \mu_X(t, X_t)dt + \sigma_X(t, X_t)dW_t, \qquad F_t(T) = h(t, T, X_t)$$
$$r_t = h(t, t, X_t)$$

  where the initial forward curve $F_0(T) = h(0, T, X_0)$ can be observed on the market.
- The HJM framework can, by construction, match the initial term structure.

Heath-Jarrow-Morton

- The dynamics of the forward rate follow from Itô's lemma:

$$\mathrm{d}F_t(T) = \mathrm{d}h(t, T, X_t) = \mu_F(t, T, X_t)\mathrm{d}t + \sigma_F(t, T, X_t)\mathrm{d}W_t$$

- Therefore,

$$F_t(T) = F_0(T) + \int_0^t \mu_F(s, T, X_s)\mathrm{d}s + \int_0^t \sigma_F(s, T, X_s)\mathrm{d}W_s$$
$$r_t = F_0(t) + \int_0^t \mu_F(s, t, X_s)\mathrm{d}s + \int_0^t \sigma_F(s, t, X_s)\mathrm{d}W_s$$

- One can show that under $\mathbb{Q}$, the drift terms are fully determined by the specification of the volatility terms $\sigma_F(t, T, X_t)$, and more precisely ...

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Heath-Jarrow-Morton Drift Condition

Assume that the induced bond market is arbitrage free. Then there exists a k-dimensional column-vector process $\lambda(t, T, X_t)$ (market price of risk) such that

$$\mu_F(t, T, X_t) = \sigma_F(t, T, X_t) \int_0^t \sigma_F(s, T, X_s)\mathrm{d}s + \sigma_F(t, T, X_t)\lambda(t, T, X_t)$$

- I skip the proof, and focus on the implications:
- $\mathbb{Q}$-dynamics of the forward curve:

$$\begin{aligned} \mathrm{d}F_t(T) &= [\underbrace{\mu_F(t, T, X_t) - \sigma_F(t, T, X_t)\lambda(t, T, X_t)}_{\mu_F^{\mathbb{Q}}(t,T,X_t)}]\mathrm{d}t + \sigma_F(t, T, X_t)\mathrm{d}W_t^{\mathbb{Q}} \\ &= \sigma_F(t, T, X_t)\Big(\int_0^t \sigma_F(s, T, X_s)\mathrm{d}s\Big)\mathrm{d}t + \sigma_F(t, T, X_t)\mathrm{d}W_t^{\mathbb{Q}} \end{aligned}$$

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- Interest Rates under $\mathbb{Q}$

$$F_t(T) = F_0(T) + \int_0^t \sigma_F(s, T, X_s)\Big(\int_0^s \sigma_F(\tau, T, X_\tau)\mathrm{d}\tau\Big)\mathrm{d}s + \int_0^t \sigma_F(s, T, X_s)\mathrm{d}W_s^{\mathbb{Q}}$$

$$r_t = F_t(t)$$

- Recipe for the HJM framework:
  1. Specify, by your own choice, the volatilities σ_F.
  2. Determine the drift rate of forward rates under $\mathbb{Q}$: $\mu_F^{\mathbb{Q}}(t, T, X_t) = \sigma_F(t, T, X_t)\int_0^t \sigma_F(s, T, X_s)\mathrm{d}s$.
  3. Go to the market and observe today's forward rate structure $F_0(T)$.
  4. Calculate or simulate the evolution of the term structure $F_t(T)$.
  5. Determine bond prices $P_t(T) = \exp(-\int_t^T F_t(s)\mathrm{d}s)$.
  6. Calculate prices of interest rate derivatives.

Problem: Special Cases

1. Suppose the forward rate volatility is given by $\sigma_F(t, T, X_t) = \sigma$. Show that this specification implies the Ho-Lee model.
2. Suppose the forward rate volatility is given by $\sigma_F(t, T, X_t) = \sigma \mathrm{e}^{-a(T-t)}$. Show that this specification implies the Hull-White model.
3. Show that if $\sigma_F(t, T, X_t)$ is a deterministic function of t and T, all short rates and forward rates are normally distributed. Besides, all bond prices are log-normally distributed.

Solution:

Problem: Special Cases

Problem: Special Cases

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- Since the seminal work of Black (1976) practitioners have been using the Black76-formula for caplets and floorlets.
- A caplet with maturity T_i and strike rate L_C on a notional N has payoff at time T_i:

$$C_{T_i} = (L_{T_{i-1}} - L_C)^+ \Delta_{T_{i-1}} N$$

where $L_{T_{i-1}}$ denotes the spot LIBOR rate for $[T_{i-1}, T_i]$.
- Black (1976) *postulates* the following pricing formula for $t \leq T_{i-1}$:

$$C_t = \Delta_{T_{i-1}} P_t(T_i) L_t(T_{i-1}, T_i) N \cdot \Phi(d_1) - P_t(T_i) \cdot L_C \Delta_{T_{i-1}} N \cdot \Phi(d_2)$$

where d_1 and d_2 are very similar to the terms in the Black-Scholes model.

The T-Forward Measure

- Recall: Numéraire-dependent pricing formula

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right].$$

- We have used
  - $\mathbb{Q}$ associated to the MMA
  - $\mathbb{P}$ associated to the numéraire portfolio
  - $\mathbb{Q}_S$ associated to the stock
- For the pricing of interest rate options, it has proven to be useful to use T-bonds with price $P_t(T)$ as numéraire.
- The corresponding EMM is the so-called T-forward measure $\mathbb{Q}_T$.

$$C_t = P_t(T) E_t^{\mathbb{Q}_T}\left[C_T\right].$$

- This measure disentangles discounting and the calculation of the expectation.

The T-Forward Measure

- Recall: Numéraire-dependent pricing formula

$$C_t = N_t E_t^{\mathbb{Q}_N}\left[\frac{C_T}{N_T}\right].$$

- We have used
  - $\mathbb{Q}$ associated to the MMA
  - $\mathbb{P}$ associated to the numéraire portfolio
  - $\mathbb{Q}_S$ associated to the stock
- For the pricing of interest rate options, it has proven to be useful to use T-bonds with price $P_t(T)$ as numéraire.
- The corresponding EMM is the so-called T-forward measure $\mathbb{Q}_T$.

$$C_t = P_t(T) E_t^{\mathbb{Q}_T}\left[C_T\right].$$

- This measure disentangles discounting and the calculation of the expectation.

Prove that under $\mathbb{Q}_T$, the instantaneous forward rate $F_0(T)$ is the expected future short rate r_T, i.e.,

$$F_0(T) = \mathbb{E}^{\mathbb{Q}_T}[r_T].$$

Solution:

LIBOR Market Model

- Model the LIBOR forward rates $L_t(T_{i-1}, T_i)$ such that they are log-normally distributed under the T_i-forward measure.
- The LIBOR market model:

$$\mathrm{d}L_t(T_{i-1}, T_i) = L_t(T_{i-1}, T_i)\sigma_i(t)'\mathrm{d}W_t^{\mathbb{Q}_{T_i}}$$

  where $\sigma_i(t) \in \mathbb{R}^k$, $W^{\mathbb{Q}_{T_i}}$ is a k-dimensional Brownian motion.
- Remark: From the definition it is not obvious that, given a specification of $\sigma_i(t)$, there exists a corresponding LIBOR market model. However, it does!
- Idea: Model all LIBOR rates under a common reference measure, the *terminal measure* $\mathbb{Q}_T$ with $T = T_n$

$$\mathrm{d}L_t(T_{i-1}, T_i) = \mu_i(t, L_t)\mathrm{d}t + L_t(T_{i-1}, T_i)\sigma_i(t)\mathrm{d}W_t^{\mathbb{Q}_T}$$

The Drift Condition

- If one chooses the drift rate appropriately, one obtains the desired LIBOR market specification

$$dL_t(T_{i-1}, T_i) = L_t(T_{i-1}, T_i)\sigma_i(t)' dW_t^{\mathbb{Q}_{T_i}}$$

- One can show that the required drift specification is

$$\mu_i(t, L_t) = -L_t(T_{i-1}, T_i) \sum_{k=i+1}^{n} \frac{\Delta_{T_{k-1}}}{1 + L_t(T_{k-1}, T_k)\Delta_{T_{k-1}}} \sigma_i(t)' \sigma_k(t),$$

$$\mu_n(t, L_t) = 0.$$

- Takeaway: We can model LIBOR rates under the *common* terminal measure $\mathbb{Q}_T$ such that LIBOR forward rates $L_t(T_{i-1}, T_i)$ are log-normally distributed martingales under "*their*" T_i-forward measure $\mathbb{Q}_{T_i}$.

Modeling Choices

- To complete the LIBOR model, one still needs to specify the number k of Brownian motions and the volatilities $\sigma_i(t)$.
- The number k is usually chosen in the range from one to three (correlation does not affect the prices of plain vanilla options, but of more complicated products).
- The volatilities $\sigma_i(t)$ are obtained by calibration to observed price data, i.e., they are *implied* volatilities to match prices of interest rate options. Dependence on time t is often allowed, to ensure sufficient flexibility. $\sigma_i(t)$ is typically a piecewise constant scalar function with jumps at the reset dates.
- Use this calibrated model to determine the prices of more complex products.
- Note: the LIBOR market model does not specify the short rate process and can only price a limited range of term structure products in closed-form.

The Black76 Formula

- Under the T_i-forward measure, the LIBOR forward rate $L_t(T_{i-1}, T_i)$ is a martingale and it is log-normally distributed. Hence, we are in a similar situation as in the Black-Scholes model.
- Straightforward calculations show that the price of a caplet is given by

$$C_t = P_t(T_i)\big[L_t(T_{i-1}, T_i) \cdot \Phi(d_1) - L_C \cdot \Phi(d_2)\big]\Delta_{T_{i-1}} N$$

where

$$d_1 = \frac{\log\left(\frac{L_t(T_{i-1}, T_i)}{L_C}\right) + \frac{1}{2}\Sigma_i(t, T_{i-1})^2}{\Sigma_i(t, T_{i-1})}$$

$$d_2 = d_1 - \Sigma_i(t, T_{i-1})$$

$$\Sigma_i(t, T_{i-1})^2 = \int_t^{T_{i-1}} \|\sigma_i(s)\|^2 \mathrm{d}s$$

Comparison to Black-Scholes

Some Remarks

- There is a one-to-one mapping between the volatility and the caplet price. There is no ambiguity in quoting the price of a caplet simply by quoting its "Black volatility" or implied volatility.
- Caps and floors have the same implied volatility for a given strike.
- As negative interest rates became a possibility, the Black model became increasingly inappropriate. Many variants have been proposed, including shifted log-normal and normal, though a new standard is yet to emerge.
- There is a very general option pricing formula for a European call option with strike K and maturity T on an underlying S. One can show that under mild assumptions the price of a European call option has always the form

$$C_t = S_t \mathbb{Q}^S(S_T > K) - P_t(T) K \mathbb{Q}^T(S_T > K).$$

where $\mathbb{Q}^S$ is an EMM that takes the underlying as numéraire, and $\mathbb{Q}^T$ is the T-forward measure.

General Option Pricing Formula

- This formula holds for *any* arbitrage-free financial market model.
- Suppose the process $\widehat{S}_t = \frac{S_t}{P_t(T)}$ satisfies a stochastic differential equation of the form

$$d\widehat{S}_t = \widehat{S}_t \mu(t, T)dt + \widehat{S}_t \sigma(t, T)dW_t,$$

- Then, the price of the call option is

$$C_t = S_t N(d_1) - P_t(T) K N(d_2)$$

with

$$d_1 = \frac{\log\left(\frac{S_t}{K P_t(T)}\right) + \frac{1}{2}\Sigma(t, T)^2}{\Sigma(t, T)}$$

$$d_2 = d_1 - \Sigma(t, T)$$

$$\Sigma(t, T)^2 = \int_t^T \|\sigma(s, T)\|^2 ds$$

1. Derive the price of a European call option on a T_2-bond with strike price K and maturity in $T_1 < T_2$ in the Hull-White model,

$$\mathrm{d}r_t = a(b^{\mathbb{Q}}(t) - r_t)\mathrm{d}t + \sigma \mathrm{d}W_t^{\mathbb{Q}}$$

2. Explain the differences between your result and the option price in the Vasicek model.

Solution:

Problem: Option Pricing in the Hull-White Mode

Swap Market Model

- The *swap market model* is a variant of the LIBOR market model.
- In the swap market model, par swap rates are modeled to be log-normally distributed, rather than LIBOR rates.
- The swap market model is commonly used to price swaptions, i.e., options on swap contracts, for which a variant of the Black76 formula exists.
- It can be shown that LIBOR market models and swap market models are incompatible, i.e., par swap rates are not log-normally distributed in the LIBOR market model, and LIBOR rates are not log-normally distributed in swap market models.

Part VII

A Brief Introduction to Credit Risk

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Motivation

- So far, we have considered discount factors and term structures related to default-free bonds.
- In reality there is always credit risk, i.e., the risk of default from an issuer of a bond (the borrower) failing to make the payments

Definition: Credit Risk

Credit risk is the risk that the holder of a financial asset experiences a loss because of

- a debtor's non-payment of a loan or other line of credit (either the principal or interest (coupon) or both)
- a default by the counterparty in a derivatives transaction.

- Credit risk differs from market risk since
  - default is a 0-1-event
  - default risk is harder to measure
  - default risk cannot be hedged away by a market index

How to Quantify Credit Risk?

- There are two dimensions of credit risk:
  1. How likely is a default?
  2. How big is the loss if a default occurs?
- These dimensions are captured by the
  1. default probability (PD),
  2. loss given default (LGD), L_τ.
- Recovery rate $R_\tau = 1 - L_\tau$
- Can these quantities be identified from historical data? For instance, BASF has never defaulted. Does this mean that its default probability is zero?
- **Idea:** Back out credit risk from the prices of credit derivatives and corporate bonds.

- We are now going to introduce discount factors corresponding to defaultable zero coupon bonds.
- Let the defaultable zero coupon bond's maturity be T and its face value be 1. Denote its value at time $t \leq T$ by $P_t^d(T)$.
- Modeling credit risk is usually done by introducing a random (first) default time $\tau \in \mathbb{R}^+$.
    - In case of no default ($\tau > T$), the bond pays off 1 at time T.
    - In case of default ($\tau \leq T$), the bond pays off $R_\tau = 1 - L_\tau$ at time T.

  Here $L_\tau \in (0, 1]$ is the loss rate.
- The default time τ is modeled as the first jump of a counting process (typically a Poisson or a Cox process) $N_t \in \mathbb{N}$, i.e.,

$$\tau = \min\{t \mid N_t = 1\}$$

Poisson and Cox Processes

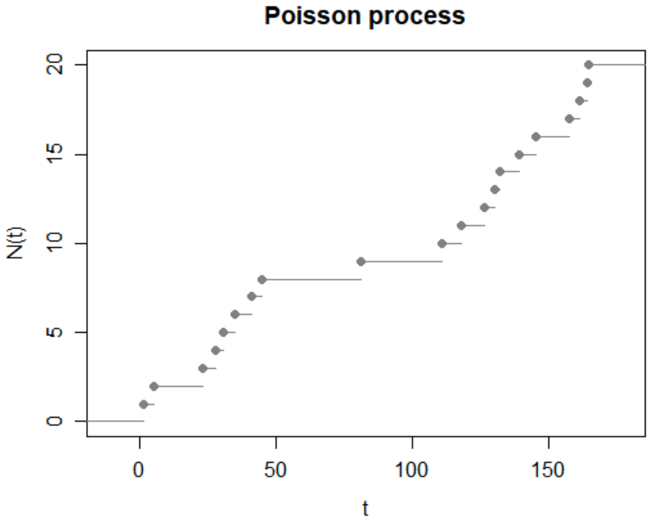
- A *Poisson process* N is an increasing process taking values in $\mathbb{N}$ (a so-called counting process) with
  1. $N_0 = 0$
  2. independent increments
  3. the number of events (or points) in any interval of length t is a Poisson random variable with mean λt.
- The parameter λ is called the jump intensity (or *default intensity*, or *hazard rate*) and models the instantaneous default probability, i.e.,

$$\lambda = \lim_{\Delta t \to 0} \frac{\mathbb{P}(N_{t+\Delta t} > N_t)}{\Delta t}$$

- If the parameter λ is itself a non-negative stochastic process, we call N a Cox process. A typical choice is that λ is of the CIR type, i.e.,

$$\mathrm{d}\lambda_t = a(b - \lambda_t)\mathrm{d}t + \sigma\sqrt{\lambda_t}\mathrm{d}W_t$$

Poisson and Cox Processes



Interpretation of Default Intensity

- Consider a Poisson process $N^{\mathbb{Q}}$ with intensity $\lambda^{\mathbb{Q}}$ under $\mathbb{Q}$. Default happens if the first jump of N happens before maturity.
- Probability of default under $\mathbb{Q}$

$$\mathbb{Q}(\tau \leq T) = \mathbb{Q}(N_T \geq 1) = 1 - \mathbb{Q}(N_T = 0) =_{(3)} 1 - \mathrm{e}^{-\lambda^{\mathbb{Q}} T}$$

- In particular, the one-year default probability is

$$\mathbb{Q}(\tau < 1) = 1 - \mathrm{e}^{-\lambda^{\mathbb{Q}}} \approx \lambda^{\mathbb{Q}}$$

- Consequently, the default intensity is approximately the one-year probability of default.
- In reality, default probabilities are not constant, but depend on macroeconomic indicators and firm-specific variables.

Basic Relations

- Standing assumption: Default intensity λ_t, short rate r_t, and recovery rate R_t are stochastically independent.
- Under this assumption, interest rate risk can be disentangled from default risk.

$$
\begin{aligned}
P_0^d(T) &= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r_s ds} 1_{\{\tau > T\}} + \mathrm{e}^{-\int_0^T r_s ds} 1_{\{\tau \leq T\}} R_\tau\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r_s ds}\right]\mathbb{Q}(\tau > T) + \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T r_s ds}\right]\mathbb{Q}(\tau \leq T)\mathbb{E}^{\mathbb{Q}}[R_\tau] \\
&= P_0(T)\Big(\mathbb{Q}(\tau > T) + \mathbb{Q}(\tau \leq T)\mathbb{E}^{\mathbb{Q}}[R_\tau]\Big) \\
&= P_0(T)\Big(1 - E^{\mathbb{Q}}[L_\tau]\mathbb{Q}(\tau \leq T)\Big) \\
&= P_0(T)\Big(1 - E^{\mathbb{Q}}[L_\tau](1 - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \lambda_s^{\mathbb{Q}} ds}\right])\Big)
\end{aligned}
$$

Credit Spread

- The credit spread between both bonds:

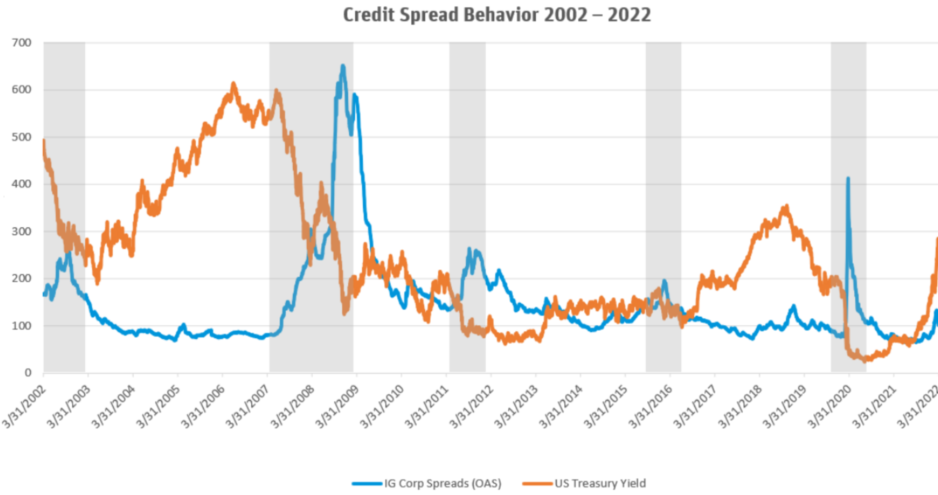
$$
\begin{aligned}
S_0^d(T) &= R_0^d(T) - R_0(T) \\
&= -\frac{1}{T}\log P_0^d(T) + \frac{1}{T}\log P_0(T) \\
&= -\frac{1}{T}\log\left(1 - E^{\mathbb{Q}}[L_\tau]\left(1 - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \lambda_s^{\mathbb{Q}} ds}\right]\right)\right) \\
&\approx \frac{1}{T}E^{\mathbb{Q}}[L_\tau]\left(1 - \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_0^T \lambda_s^{\mathbb{Q}} ds}\right]\right)
\end{aligned}
$$

- If the default intensity λ is constant:

$$
\begin{aligned}
S^d(T) &\approx \frac{1}{T}\mathbb{E}^{\mathbb{Q}}[L_\tau](1 - \mathrm{e}^{-\lambda^{\mathbb{Q}}T}) \\
&\approx \lambda^{\mathbb{Q}}\mathbb{E}^{\mathbb{Q}}[L_\tau]
\end{aligned}
$$

- **Rule of thumb:** Yield spread between corporate bond and Treasury bond approximately equals the expected one-year loss due to default risk under the risk-neutral measure.

Credit Spread



Some Remarks

- A thorough quantitative analysis of credit risk requires Itô calculus with jump processes.
- Term structure equations become more complicated as they involve jump terms.
- If both the short rate process and the intensity process are affine, then the corporate bond prices before default are affine as well, i.e.,

$$P_t^d(T)1_{\{t<\tau\}} = \mathrm{e}^{A^d(t,T)+B^d(t,T)r_t+C^d(t,T)\lambda_t}$$

- Jump processes are also commonly used to model stock market crashes. A simple example is the Merton Jump-Diffusion model

$$\mathrm{d}S_t = S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t + S_t\ell_t\mathrm{d}N_t.$$

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Idea: Merton's Firm Value Model

Merton's Firm Value Model

Tilburg University
School of Economics and Management

- Firm has debt – modeled by a zero bond with
  - notional F
  - maturity at time T
  - default only at time T possible
- At T: Redemption depends on the firm value V_T

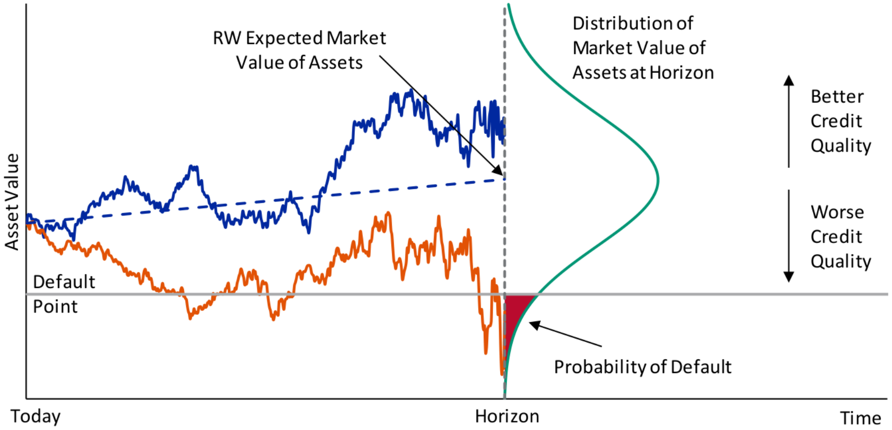
$$D_T = \min\{V_T, F\}$$

  If $V_T < F$: default.
  $\Longrightarrow$ Loss given default: $L = F - V_T$
- Shareholders get the residuum

$$\begin{aligned} E_T &= V_T - D_T \\ &= V_T - \min\{V_T, F\} \\ &= \max\{V_T - F, 0\} \end{aligned}$$

  $\Longrightarrow$ Equity is a call option on the firm value with maturity at time T and strike price F.

Merton's Firm Value Model



Source: Moody's Research Analytics

Merton's Firm Value Model

- Model the firm value like the stock price in the Black-Scholes model (V is log-normally distributed)
- Equity is a call option on the firm value
  $\Longrightarrow$ Black-Scholes formula delivers:

$$
\begin{aligned}
E_0 &= V_0\Phi(d_1) - Fe^{-rT}\Phi(d_2) \\
D_0 &= V_0 - E_0 = Fe^{-R^d(T)T} \\
d_1 &= \frac{\ln(V_0/F) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}
\end{aligned}
$$

- Credit spread:

$$S_0(T) = \frac{1}{T}\log\left(\frac{F}{D_0}\right) - r$$

- Weaknesses
  - Same weaknesses as the Black-Scholes model (e.g., constant volatility, interest rates)
  - V is typically not traded (but E).$\Longrightarrow$ How do we know σ?

$$\sigma \frac{\Phi\big(d_1(\sigma)\big)}{E(\sigma)} = \frac{\sigma_E}{V}$$

  - Very simplistic debt policy. Firms do not emit just one zero bond. In reality, they emit several coupon bonds, mortgages, and other forms of credit contracts with different maturities.
- However, economic implications are quite plausible.
- Firm value model acts as a building block for many practically-relevant models (e.g., Moody's KMV Model, J.P. Morgans' Credit Metrics, ...)
- Popular alternative model in credit risk management: Credit Risk+