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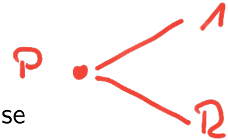
Reduced-form Modeling vs. Structural Modeling

Intensity models



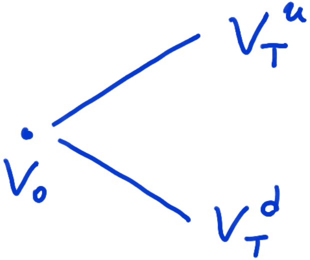
Reduced-form models

- Default happens as a sudden surprise
- Well-suited to determine CDS prices etc. and for portfolio optimization
- But information is disregarded (stock and option prices)
- A default is sometimes preceded by a period of a declining stock price
  $\Rightarrow$ No information why default happens!

Alternative: Firm value models (syn. structural models)

- Explicitly model the link between equity and debt
- Taylor-made for management purposes but more complicated
  $\Rightarrow$ Incorporate information why default happens!

Assets	Liabilities
A	Equity E_T Debt $D_T = F$
$A = V$	$E + D = V$



Only one zero-coupon bond with face value F that matures at T

$$V_T^u > F > V_T^d$$

$$E_T + D_T = V_T$$

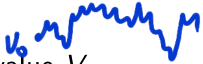
$$E_T = V_T - D_T = V_T - \min\{V_T; F\}$$

$$= \max\{V_T - F; 0\}$$

Merton's Firm Value Model

- Firm has debt – modeled by a zero bond with
  - notional F
  - maturity at time T
  - default only at time T possible

V_t follows a GBM



- At T: Redemption depends on the firm value V_T

$$D_T = \min\{V_T, F\}$$

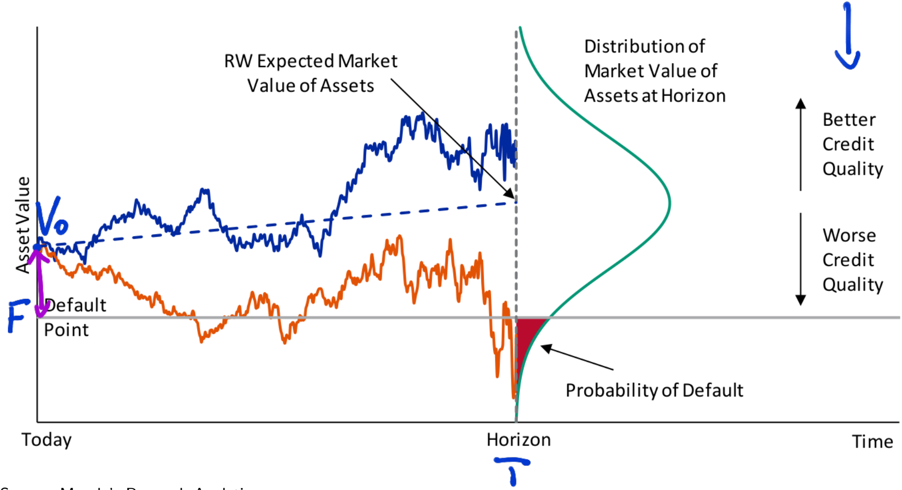
If $V_T < F$: default.
$\Longrightarrow$ Loss ~~given default~~: $L = F - V_T$

- Shareholders get the residuum

$$\begin{aligned} E_T &= V_T - D_T \\ &= V_T - \min\{V_T, F\} \\ &= \max\{V_T - F, 0\} \end{aligned}$$

$\Longrightarrow$ Equity is a call option on the firm value with maturity at time T and strike price F.

Merton's Firm Value Model



Source: Moody's Research Analytics

Merton's Firm Value Model: Solution

- Model the firm value like the stock price in the Black-Scholes model (V is log-normally distributed) and evolves according to

$$dV_t = V_t[\mu dt + \sigma dW_t].$$

- Equity is a call option on the firm value
$\Longrightarrow$ Black-Scholes formula delivers:

$S_0 \phi(d_1) - K e^{-rT} \phi(d_2)$

$E_0 + D_0 = V_0$

$$\begin{aligned} E_0 &= V_0\Phi(d_1) - Fe^{-rT}\Phi(d_2) \\ D_0 &= V_0 - E_0 = Fe^{-y^d T} \\ d_{1,2} &= \frac{\ln(V_0/F) + (r \pm 0.5\sigma^2)T}{\sigma\sqrt{T}} \end{aligned}$$

$\Rightarrow y^d = -\frac{1}{T}\log\left(\frac{D_0}{F}\right)$

- Credit spread:

$y^d - r$

$$s_0(T) = \frac{1}{T}\log\left(\frac{F}{D_0}\right) - r$$

Merton's Firm Value Model: Default Probabilities

- A default only occurs at maturity T if and only if the notional of the bond exceeds the firm value, i.e.,

$$\mathbb{P}(\text{default at time } T) = \mathbb{P}(V_T < F)$$

- Since the firm value is a geometric Brownian motion, it can be written as

$$V_T = V_0 \mathrm{e}^{(\mu - 0.5\sigma^2)T + \sigma W_T},$$

where $W_T \sim \mathcal{N}(0, T)$, i.e., with $Z \sim \mathcal{N}(0, 1)$ $W_T = \sqrt{T} \cdot Z$

$$V_T = V_0 \mathrm{e}^{(\mu - 0.5\sigma^2)T + \sigma\sqrt{T}Z}.$$

Merton's Firm Value Model: Default Probabilities

- Consequently, the default probability can be calculated

$$\begin{aligned}
\mathbb{P}(\text{default at time } T) &= \mathbb{P}(V_T < F) \\
&= \mathbb{P}\Big(V_0 \mathrm{e}^{(\mu - 0.5\sigma^2)T + \sigma\sqrt{T}Z} < F\Big) \\
&= \mathbb{P}\Big(Z < \frac{\log(F/V_0) - (\mu - 0.5\sigma^2)T}{\sigma\sqrt{T}}\Big) \\
&= \Phi\Big(\frac{\log(F/V_0) - (\mu - 0.5\sigma^2)T}{\sigma\sqrt{T}}\Big) \\
&= 1 - \Phi\Big(\underbrace{\frac{\log(V_0/F) + (\mu - 0.5\sigma^2)T}{\sigma\sqrt{T}}}\Big) \\
&= 1 - \mathbb{P}(\text{no default at time } T)
\end{aligned}$$

Φ is CDF of $\mathcal{N}(0,1)$

- Same calculations under $\mathbb{Q}$:

$$\mathbb{Q}(\text{default at time } T) = 1 - \Phi(d_2).$$

not d_2 but similar because it uses μ instead of r

Merton's Firm Value Model: Discussion

- Weaknesses
  - Same weaknesses as the Black-Scholes model (e.g., constant volatility, interest rates).
  - V is typically not traded (but E) $\Longrightarrow$ How do we know σ and V?

$$E = V\Phi(d_1) - Fe^{-rT}\Phi(d_2), \qquad \sigma\frac{\Phi(d_1)}{E} = \frac{\sigma_E}{V}$$

  - Very simplistic debt policy. Firms do not emit just one zero bond. In reality, they emit several coupon bonds, mortgages, and other forms of credit contracts with different maturities.
- However, economic implications are quite plausible.
- Firm value model acts as a building block for many practically-relevant models (e.g., Moody's KMV Model, J.P. Morgans' CreditMetrics, ...).
- Popular alternative model in credit risk management: Credit Risk+

The KMV Model in a Nutshell

- KMV overcomes the debatable debt policy in Merton's model by introducing short-term liabilities and long-term debt.
- It also replaces the normality assumption of asset returns by an empirical distribution.
- Empirically, firms default when the asset value reaches a level somewhere between the value of total liabilities and the value of short-term debt.
- KMV computes an index called "distance-to-default" (DD) defined as the number of standard deviations between the mean of the distribution of the asset value, and the "default point".
- The default point is set at the par value of current liabilities including short-term debt to be serviced over the time horizon, plus half the long-term debt.

The distance to default

- We use the following notation:
  - *STD*: short-term debt
  - *LTD*: long-term debt
  - *DPT*: default point
  - *DD*: distance-to-default: distance between the expected asset value in 1 year and the default point expressed in standard deviation of future asset returns.
- In Merton's model, the probability of default is

$$\mathbb{P}(\text{default at time } 1) = 1 - \mathbb{P}\Big(Z < \underbrace{\frac{\log(V_0/F) + (\mu - 0.5\sigma^2)}{\sigma}}_{=DD}\Big)$$

  where $Z \sim \mathcal{N}(0,1)$ under $\mathbb{P}$.
- *DD* measures how far away the firm is from a default.

From Merton to KMV

- KMV replaces that expression by

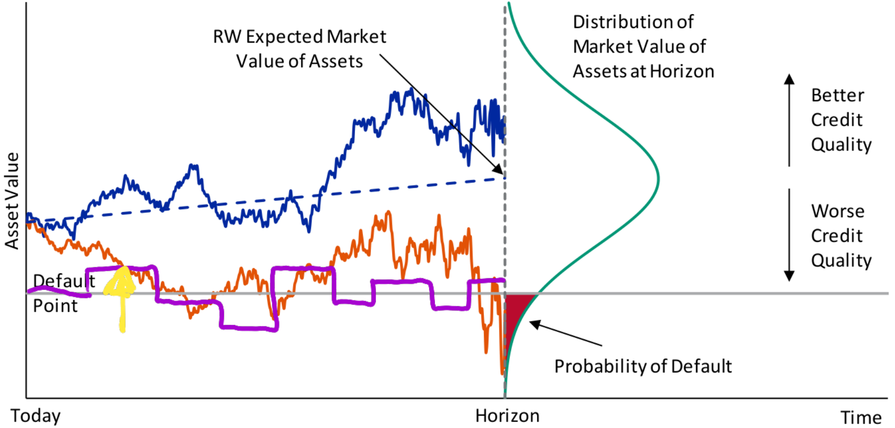
$$DD = \frac{\mathbb{E}[V_1] - DPT}{\sigma V_0}$$

  where $DPT = STD + \frac{1}{2}LTD$, and Z is no longer normally distributed.
- This seems to be a bit ad-hoc (and it is!), but it can be motivated.

$$DD_{Merton} = \frac{\log(V_0/F) + (\mu - 0.5\sigma^2)}{\sigma} \approx \frac{\log(\mathbb{E}[V_1]/F)}{\sigma} \approx \frac{\mathbb{E}[V_1] - F}{\sigma V_0}$$

- Now, replace the debt policy by a more realistic policy, i.e., F by DPT, and adjust the probability distribution of asset returns.
- In particular, KMV assumes that DPT is itself random.

Illustration



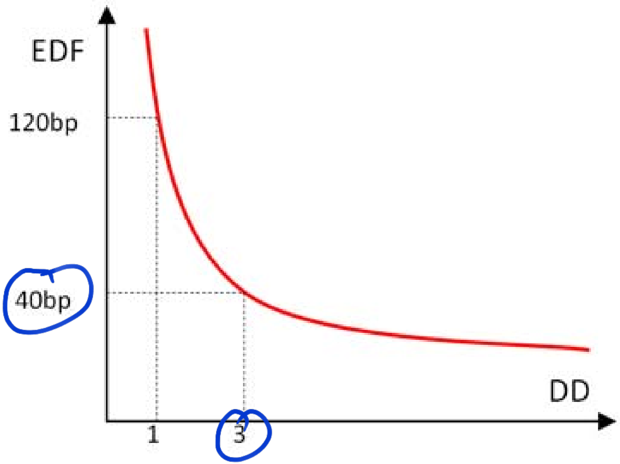
Expected Default Frequencies

- The KMV model replaces the normality assumption of asset returns by an empirical distribution that maps *DD* to physical default probabilities.
- KMV calls these probabilities *Expected Default Frequencies* (EDF).
- Obviously, *DD* and EDF are inversely related to each other.
- Consider the following **example**: $DPT = 600 + \frac{1}{2} \cdot 400 = 800$
  - $V_0 = 1000$, $\widehat{\mu} = 10\%$, $STD = 600$, $LTD = 400$, $\sigma = 10\%$. $\mathbb{E}[V_1] = 1100$, $DPT = 800$, $DD = \frac{1100-800}{100} = 3$.
  - Assume that among the population of all the firms with a *DD* of 3 at one point in time, say 5000 firms, 20 defaulted 1 year later, then:

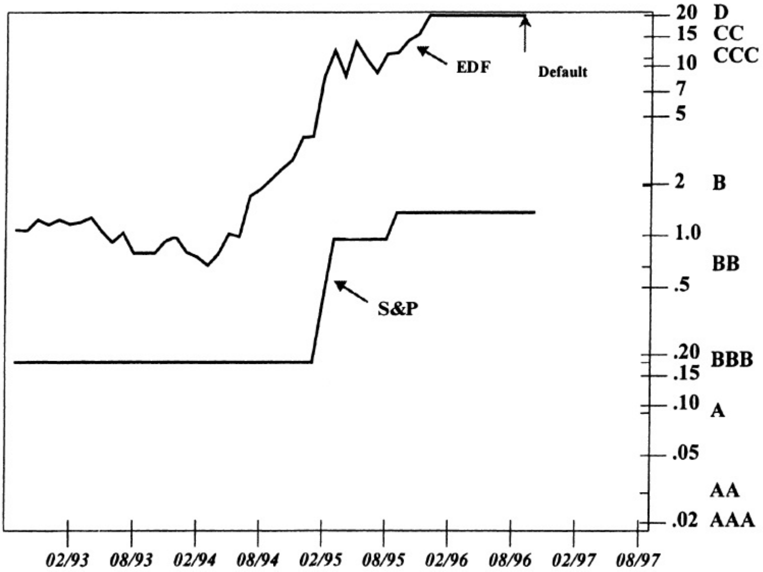
$$EDF(DD = 3) = \frac{20}{5000} = 40bps$$

  - Hence, the EDF as a proxy for the default probability under $\mathbb{P}$ is 0.4%, which roughly corresponds to a BB^+ rating.

Expected Default Frequencies



EDF as a Predictor of Default



Some Remarks

- We can also derive the *term structure of EDFs*, i.e., the EDF as a function of the maturity of the bond.
- The KMV model is widely used in practice for risk management purposes. In particular, it is used to determine the VaR and CVaR of credit portfolios.
- For pricing purposes, it is important to know the risk-neutral default probabilities rather than the physical ones.
- It is also possible to derive the risk-neutral EDFs which act as a proxy for the risk-neutral default probabilities. These default probabilities can then be used to disentangle the two dimensions of credit risk (probability of default and LGD) under the $\mathbb{Q}$ measure since credit spreads are directly observable.

CreditMetrics in a Nutshell

- CreditMetrics estimates the forward distribution of the changes in value of a portfolio of loan and bond type products at a given time horizon.
- The method takes contagion effects into account.
- By contrast to KMV, the default probabilities are not changing continuously, but CreditMetrics relies upon credit ratings.
- Transitions between ratings are taken into account, i.e., the default probability changes dynamically.
- Besides pricing, it can be used for risk management purposes to determine the VaR and CVaR for credit portfolios.
- The method can be applied to single assets and whole portfolios of loans (Monte-Carlo simulation needed).

Procedure for a Single Loan

1. Choose a rating system (e.g., Moody's, Fitch, S&P) including
   - Rating categories
   - Transition probability matrix

   All issuers are assumed to be homogeneous within a given category.
2. Specify the time horizon (e.g., 1 year).
3. Specify the forward discount curve at the time horizon for each credit category, and, in the case of default, the recovery rate of the instruments under consideration.
4. Translate this information into the forward distribution of the changes in portfolio value consecutive to credit migration.

Example: BBB-bond maturing in 5 years, coupon of 6%

1. Choose a rating system: transition probabilities

Initial Rating	Rating at year-end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81	8.33	0.68	0.06	0.12	0	0	0
AA	0.70	90.65	7.79	0.64	0.06	0.14	0.02	0
A	0.09	2.27	91.05	5.52	0.74	0.26	0.01	0.06
BBB	0.02	0.33	5.95	86.93	5.30	1.17	0.12	0.18
BB	0.03	0.14	0.67	7.73	80.53	8.84	1.00	1.06
B	0	0.11	0.24	0.43	6.48	83.46	4.07	5.20
CCC	0.22	0	0.22	1.30	2.38	11.24	64.86	19.79

Source: Standard & Poor's CreditWeek April 15, 1996

Example: BBB-bond Maturing in 5 Years, Coupon of 6%

1. Choose a rating system: cumulative default probabilities

Average cumulative default rates (%)[a]

Term	1	2	3	4	5...	7...	10...	15
AAA	0.00	0.00	0.07	0.15	0.24...	0.66...	1.40...	1.40
AA	0.00	0.02	0.12	0.25	0.43...	0.89...	1.29...	1.48
A	0.06	0.16	0.27	0.44	0.67...	1.12...	2.17...	3.00
BBB	*0.18*	*0.44*	*0.72*	*1.27*	*1.78...*	*2.99...*	*4.34...*	*4.70*
BB	1.06	3.48	6.12	8.68	10.97...	14.46...	17.73...	19.91
B	5.20	11.00	15.95	19.40	21.88...	25.14...	29.02...	30.65
CCC	19.79	26.92	31.63	35.97	40.15...	42.64...	45.10...	45.10

[a] *Source*: Standard & Poor's CreditWeek (April 15, 1996).

2. Specify the time horizon: assume 1 year

Example: BBB-bond Maturing in 5 Years, Coupon of 6%

3. Specify the forward discount curve at the time horizon.
   $\longrightarrow$ this step requires an interest rate risk model.
   $\longrightarrow$ see Valuation and Risk Management.

One-year forward zero-curves for each credit rating (%)[a]

Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

[a] *Source*: CreditMetrics, JP Morgan.

The 1-year forward price of the bond, if the obligor stays BBB, is then:

$$P_1 = 6 + \frac{6}{1.041} + \frac{6}{1.0467^2} + \frac{6}{1.0525^3} + \frac{106}{1.0563^4} = 107.55$$

Example: BBB-bond Maturing in 5 Years, Coupon of 6%

- Repeating the calculations for all possible end-of-year ratings yields:

One-year forward values for a BBB bond[a]

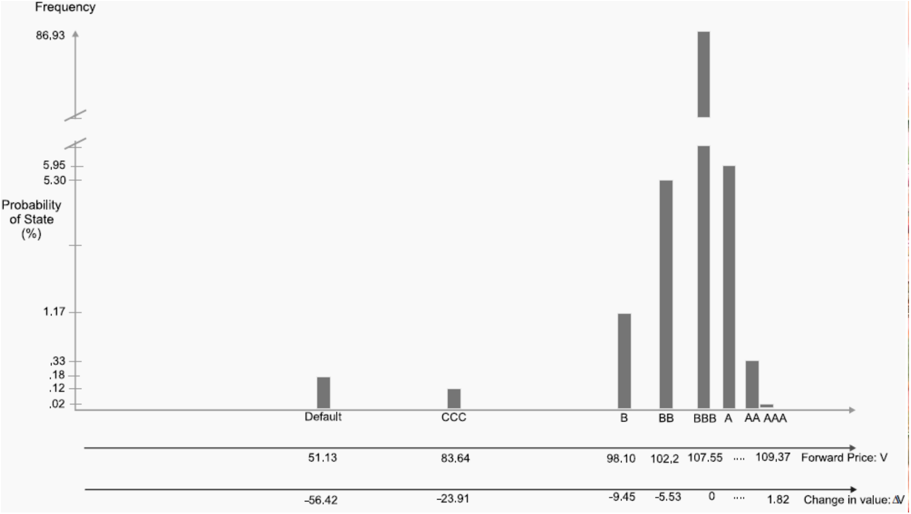
Year-end rating	Value ($)
AAA	109.37
AA	109.19
A	108.66
BBB	*107.55*
BB	102.02
B	98.10
CCC	83.64
Default	51.13

[a] *Source*: CreditMetrics, JP Morgan.

- Calculation for the default category requires an assumption about the LGD under $\mathbb{P}$. Here, $LGD = 48.87\%$ (Carty and Liebermann 1996).

Example: BBB-bond Maturing in 5 Years, Coupon of 6%

4. Translate this information into the forward distribution of the changes in portfolio value consecutive to credit migration.



Correlated Defaults in a Loan Portfolio

- So far, we have not seen that the model really builds upon the Merton model. Every ingredient seems to be endogenously given.
- Now we are focusing on a loan portfolio and have to model correlated defaults and correlated transitions.
- If transitions were uncorrelated, we could simply multiply the transition probabilities.

$$PD_{1,2} = PD_1 \cdot PD_2$$

$$BB, A: \quad P_{BB}(BB) \cdot P_A(A) = P_{BB,A}(BB, A)$$

- Instead we support the model by some Merton-type structure and construct a framework for correlated transitions.
- The Merton model is extended by CreditMetrics to include correlated changes in credit quality.
- Additional input factor is the correlation between equity returns.

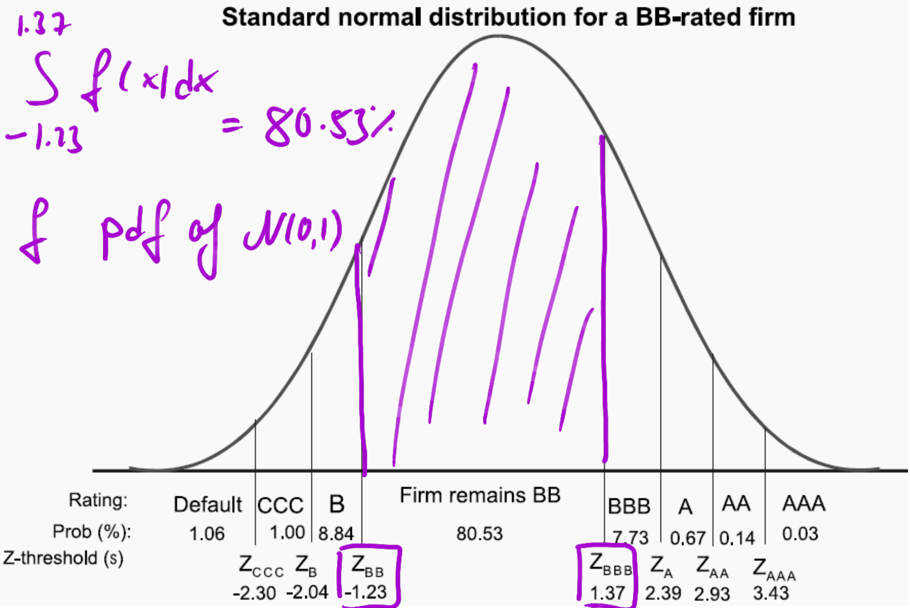
Slicing the Distribution of Asset Returns

- Recall: probability of default in Merton's model: $\Phi(-d_2^{\mathbb{P}})$

$$PD = \mathbb{P}(V_T < F) = \Phi\left(\frac{\log(F/V_0) - (\mu - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right)$$

- Idea: Replace F by various thresholds ($Z_{AAA}, Z_{AA}, Z_A, Z_{BBB}, Z_{BB}$, Z_B, Z_{CCC}) that slice the distribution of asset returns into bands such that, if we draw randomly from this distribution, we reproduce exactly the migration frequencies shown in the transition matrix.
- Z_{CCC} is the threshold point in the standard normal distribution corresponding to the probability of default.
- Note that only the threshold levels are necessary to derive the joint migration probabilities, and they are calculated without the need to observe the asset value, and to estimate its mean and variance.
- Remark: CreditMetrics uses equity data to calibrate the model, not firm value data.

Slicing the Distribution of Asset Returns



Correlated Credit Migration

- In this model, the normalized equity return for an asset in rating class R is

$$r_t^{(R)} = \frac{\log(V_t/V_0) - (\mu^{(R)} - 0.5(\sigma^{(R)})^2)T}{\sigma^{(R)}\sqrt{T}} \sim \mathcal{N}(0,1)$$

- Now, assume that equity returns are not idiosyncratic, but correlated.
- Assume that the correlation between asset rates of return is known, and is denoted by ρ. The normalized log-returns on both assets follow a joint normal distribution with density:

$$f(r^{(A)}, r^{(B)}) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left[(r^{(A)})^2 + 2\rho r^{(A)} r^{(B)} + (r^{(B)})^2\right]\right)$$

$$R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \qquad \rho^2 + c^2 = 1 \quad \Rightarrow \quad c = \sqrt{1-\rho^2}$$

Example: Credit Migration

- Determine the probability that two bonds with ratings A and BB, respectively, both stay in their rating:
- First, we have to identify the thresholds:
  - for the BB-rated bond we have $Z_B^{(BB)} = -1.23$ and $Z_{BBB}^{(BB)} = 1.37$
  - for the A-rated bond we have $Z_{BBB}^{(A)} = -1.51$ and $Z_{AA}^{(A)} = 1.98$
- Assume a correlation of $\rho = 0.2$:

$$
\begin{aligned}
&\mathbb{P}\left(Z_B^{(BB)} < r^{(BB)} < Z_{BBB}^{(BB)}, Z_{BBB}^{(A)} < r^{(A)} < Z_{AA}^{(A)}\right) \\
= &\int_{-1.23}^{1.37}\int_{-1.51}^{1.98} f(r^{(A)}, r^{(BB)}) \mathrm{d}r^{(A)} \mathrm{d}r^{(BB)} \\
= &0.7365
\end{aligned}
$$

- Correlation matters: if $\rho = 0$, this probability is 0.7332.

$P_A(A) \cdot P_{BB}(BB) = 0.7332$

Correlated Defaults

- Consider two obligors whose probabilities of default are PD_1 and PD_2, respectively. Assume an asset correlation ρ.
- Joint default probability $PD_{1,2}$:

$$
\begin{aligned}
PD_{1,2} =& \mathbb{P}(r_1^{(1)} \leq Z_{CCC}^{(1)}, r_1^{(2)} \leq Z_{CCC}^{(2)}) \\
=& \int_{-\infty}^{Z_{CCC}^{(1)}} \int_{-\infty}^{Z_{CCC}^{(2)}} f(r^{(1)}, r^{(2)}) \mathrm{d}r^{(1)} \mathrm{d}r^{(2)}
\end{aligned}
$$

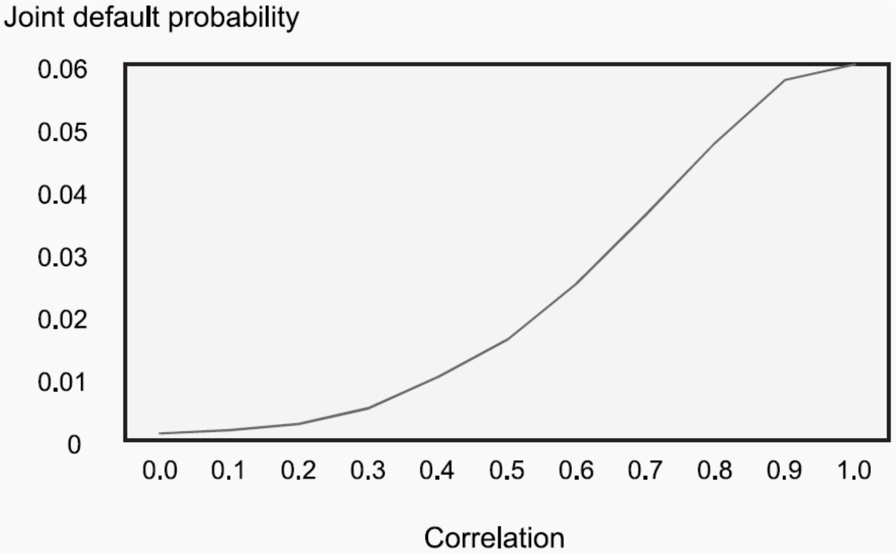
- The correlation between the two default events is thus

$E[D_1D_2] \quad E[D_1]\,E[D_2]$

$$
\rho_D = \frac{PD_{1,2} - PD_1 PD_2}{\sqrt{PD_1(1-PD_1)PD_2(1-PD_2)}} \qquad \frac{Cov_{1,2}}{\sigma_1 \sigma_2}
$$

- The ratio of asset returns correlations to default correlations is approximately 1 to 10 for asset correlations in the range of 20 to 60%. This shows that the joint probability of default is in fact quite sensitive to pairwise asset return correlations.

Example: Correlated Defaults for BB and A-rated Bonds



Large Portfolios of Bonds

- The analytical approach outlined before is not feasible for large portfolios of bonds.
- Instead, estimating the joint default probabilities and correlations is typically done by a Monte-Carlo simulation:
  1. Derivation of the asset return thresholds for each rating category.
  2. Estimation of the correlation between each pair of obligor's asset returns.
  3. Generation of asset return scenarios according to their joint normal distribution (requires Cholesky decomposition). Each scenario is characterized by n standardized asset returns.
  4. For each scenario, and for each obligor, the standardized asset return is mapped into the corresponding rating (see step 1).
  5. Given the spread curves which apply for each rating, the portfolio is revalued.
  6. Repeat the procedure a large number of times, say 100 000 times, and plot the distribution of the portfolio values.

Large Portfolios of Bonds