Agenda

Why General Equilibrium?

- So far, we have taken asset returns as given and derived prices of derivative claims.
- A general equilibrium model seeks to explain the asset pricing dynamics.
- Empirical findings (e.g., Bansal and Yaron 2004)
  - expected return on government bonds $\mathbb{E}[r_f] = 0.86\%$
  - standard deviation of return on government bonds: $\sigma(r_f) = 0.97\%$
  - equity risk premium: $\mathbb{E}[r_m - r_f] = 6.33\%$
  - standard deviation of equity return: $\sigma(r_m) = 19.42\%$
- These and other facts must be explained and not just taken as given.

General Equilibrium

- An **equilibrium** consists of
  - consumption and investment decision of each investor
  - prices of all traded assets

  such that
  - each investor maximizes his utility
  - markets clear (demand $=$ supply)
- In equilibrium, asset returns materialize **endogenously** from supply and demand.

General Equilibrium: First Outline

1. Assume that prices are given
2. Solve portfolio planning problem for each investor result: optimal portfolio holdings and optimal consumption (partial equilibrium)
3. Check market clearing condition
   - aggregate demand for assets = aggregate supply
   - aggregate consumption = aggregate endowment
4. Choose asset prices such that markets clear.

Representative Investor

- Aggregate demand depends on
  - individual demand of each investor
  - which depends on initial endowment, preferences, beliefs, ...
- Aggregation might be rather involved (we will deal with heterogeneous investors and consumption sharing rules later)
- Easy solution and standard approach: representative investor
  - one investor who represents the market
  - equilibrium condition: representative investor has to consume aggregate endowment

Racall: CAPM

Agenda

Representative Agent

- Planning horizon T.
- Representative agent
  - decides on consumption $C = (C_t)_{t=0,\dots,T}$ and investment $\varphi = (\varphi_t)_{t=0,\dots,T}$ at time $t = 0, \dots, T$.
  - gains additive utility from the consumption stream; utility index

$$\mathcal{U} = \sum_{t=0}^{T} \mathrm{e}^{-\delta t} \mathbb{E}\big[u(C_t)\big]$$

  - δ is the pure rate of time preferences.
  - u is a utility function with $u'(C) > 0$ (agents are greedy) and $u''(C) < 0$ (utility saturates).
  - Popular Choice: Power Utility (CRRA) $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$

Popular Choice: CRRA-utility

Popular Choice: CRRA-utility

- Popular utility function

$$u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$$

- The degree of relative risk aversion (Arrow 1970; Pratt 1966) is given by

$$\begin{aligned} \mathrm{RRA} &= -C \frac{u''(C)}{u'(C)} \\ &= -C \frac{-\gamma C^{-\gamma-1}}{C^{-\gamma}} \\ &= \gamma. \end{aligned}$$

- For $\gamma = 1$, the preferences collapse to log-utility $u(C) = \log(C)$ which often leads to closed-form solutions.
- However, empirical evidence suggests $\gamma > 1$.

Proof – Log-utility

More General: HARA-utility

Agenda

Consumption-based CAPM (CCAPM) – Model Setup

- To understand the mechanics, we consider a one-period model with one risk-free asset with price S_t and return r_t^f.
- The agents optimization problem is then

$$\max_{C,\varphi} \left\{ u(C_t) + \mathrm{e}^{-\delta} \mathbb{E}_t \big[u(C_{t+1}) \big] \right\}$$

subject to

$$C_t = E_t - \varphi S_t$$
$$C_{t+1} = E_{t+1} + \varphi S_t (1 + r_t^f)$$

Consumption-based CAPM (CCAPM) – Euler Condition I

- To solve the decision problem, we consider its Lagrangian

$$\begin{aligned}\mathcal{L}(C_t, C_{t+1}, \varphi, \lambda_t, \lambda_{t+1}) = u(C_t) + \mathbb{E}_t[\mathrm{e}^{-\delta} u(C_{t+1})] \\ - \lambda_t[C_t - E_t + \varphi S_t] - \lambda_{t+1}[C_{t+1} - E_{t+1} - \varphi S_t(1 + r_t^f)]\end{aligned}$$

- Taking first-order conditions

$$\begin{aligned}\mathcal{L}_{C_t} &= u_c(C_t) - \lambda_t = 0 \\ \mathcal{L}_{C_{t+1}} &= \mathbb{E}_t[\mathrm{e}^{-\delta} u_c(C_{t+1})] - \lambda_{t+1} = 0 \\ \mathcal{L}_{\varphi} &= -\lambda_t + \lambda_{t+1}(1 + r_t^f) = 0\end{aligned}$$

- Therefore,

$$\lambda_t = u_c(C_t), \qquad \lambda_{t+1} = \frac{u_c(C_t)}{1 + r_t^f}$$

Consumption-based CAPM (CCAPM) – Euler Condition II

- This implies

$$\mathbb{E}_t[\mathrm{e}^{-\delta} u_c(C_{t+1})] = \frac{u_c(C_t)}{1 + r_t^f}.$$

- We end up with the pricing equation for the risk-free asset:

Euler Condition

$$\mathbb{E}_t\left[\mathrm{e}^{-\delta} \frac{u_c(C_{t+1})}{u_c(C_t)}(1 + r_t^f)\right] = 1.$$

- This can also be achieved in multi-period settings and with more than one asset but requires the Bellman-principle (exercise!).
- Therefore, the pricing kernel is given by

$$M_{t,t+1} = \mathrm{e}^{-\delta} \frac{u_c(C_{t+1})}{u_c(C_t)}.$$

where $M_{t,t+1} = M_{t+1}/M_t$.

Consumption-based CAPM (CCAPM) – Asset Pricing

- Risk-free rate

$$\frac{1}{1+r_t^f} = \mathbb{E}_t\left[\mathrm{e}^{-\delta}\frac{u_c(C_{t+1})}{u_c(C_t)}\right].$$

- Risk premium

$$\mathrm{rp}_t = \mathbb{E}_t[r_{t+1}] - r_t^f = -(1+r_t^f)\mathrm{cov}_t[M_{t,t+1}, r_{t+1}].$$

- Typically, marginal utility is negatively correlated with asset returns.
  - Investors demand a positive risk premium to hold this asset.
  - Asset is positively correlated with consumption.
  - The opposite is true if asset returns are positively correlated with MU.

Proof – Risk Premium

Agenda

Notational Convention

- We use capital letters for a process or variable, e.g., C_t stands for consumption, D_t for dividends,...
- We use small letters for the logarithm of this variable, e.g., $c_t = \ln C_t$.
- We use the operator Δ for the time increment, e.g., $\Delta C_{t+1} = C_{t+1} - C_t$
- Notice that for any process X, we have

$$X_{t+1} = X_t \mathrm{e}^{\Delta x_{t+1}}.$$

Example: Log-Normal Case

- Investor with CRRA utility $u(C) = \frac{1}{1-\gamma} C^{1-\gamma}$, $\gamma \neq 1$.
- Assumption: Log consumption growth is normally distributed

$$\Delta c_{t+1} = \mu_c + \nu_{t+1}$$

  where μ_c denotes expected consumption growth and $\nu_{t+1} \sim_{i.i.d.} \mathcal{N}(0, \sigma_c^2)$.
- From the Euler condition, we obtain the pricing kernel

$$M_{t,t+1} = \mathrm{e}^{-\delta} \frac{u_c(C_{t+1})}{u_c(C_t)} = \mathrm{e}^{-\delta - \gamma \Delta c_{t+1}}$$

Example: Log-Normal Case

- Risk-free rate

$$e^{-r_t^f} = \mathbb{E}_t[M_{t,t+1}] = \mathbb{E}_t\left[e^{-\delta-\gamma\Delta c_{t+1}}\right]$$

Therefore,

$$r_t^f = \delta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2$$

  - δ represents the role of discounting.
  - $\gamma\mu_c$ represents intertemporal consumption smoothing.
  - $-\frac{1}{2}\gamma^2\sigma_c^2$ represents precautionary savings.
- Risky asset with i.i.d. returns $r_{t+1} \sim \mathcal{N}(\mu_i, \sigma_i^2)$

$$\mathbb{E}_t\left[e^{-\delta-\gamma\Delta c_{t+1}+r_{t+1}}\right] = 1$$

and thus

$$\mathrm{rp}_t^i = \mu_i + \frac{1}{2}\sigma_i^2 - r_t^f = \gamma\mathrm{cov}(\Delta c_{t+1}, r_{t+1}) = \gamma\sigma_{c,i}.$$

Proof: Risk-free Rate

Proof: Equity Premium

Price-Dividend Ratio

- Consumption claim X is an asset paying consumption as dividends.
- It equals financial wealth of the representative investor.
- For its price, we get

$$X_t = \mathbb{E}_t\big[M_{t,t+1}(X_{t+1} + C_{t+1})\big]$$

- Solve for the price-dividend ratio $Z_t = X_t/C_t$ which is constant in this model:

$$\begin{aligned}
X_t = ZC_t &= \mathbb{E}_t\big[M_{t,t+1}(ZC_{t+1} + C_{t+1})\big] \\
Z &= \mathbb{E}_t\Big[M_{t,t+1}(Z+1)\frac{C_{t+1}}{C_t}\Big] \\
\frac{Z}{1+Z} &= \mathbb{E}_t\big[M_{t,t+1}\mathrm{e}^{\Delta c_{t+1}}\big] \\
\frac{Z}{1+Z} &= \mathbb{E}_t\big[\mathrm{e}^{-\delta-(\gamma-1)\Delta c_{t+1}}\big] \\
\frac{Z}{1+Z} &= \mathrm{e}^{-\delta-(\gamma-1)\mu_c+\frac{1}{2}(\gamma-1)^2\sigma_c^2}
\end{aligned}$$

Price-Dividend Ratio

Agenda

Calibration

- The aim of asset pricing is to find a consumption-based explanation for asset pricing moments.
- Idea: Take $\mu_c \approx 0.02$, $\mu_i \approx 0.07$, $\sigma_c \approx 0.02$, $\sigma_i \approx 0.16$, $\rho_{c,i} \approx 0.2$ and calibrate the preferences.
- Risk-premium:

$$\begin{aligned}\gamma = \frac{\mathrm{rp}_t^i}{\sigma_{c,i}} &= \frac{\mu_i + \frac{1}{2}\sigma_i^2 - r_t^f}{\sigma_c \sigma_i \rho_{c,i}} \\ &= \frac{0.06 + 0.5 \cdot 0.16^2}{0.02 \cdot 0.16 \cdot 0.2} > 1,000\end{aligned}$$

- Experimental-based estimates of risk aversion in the range from 2 to 10.
- Moral: To match the risk-premium one needs an unrealistically high degree of risk-aversion

Even worse!

- Let's try to match the risk-free rate:

$$r_t^f = \delta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2 \approx \delta + \gamma\mu_c$$

- Then,

$$\gamma \approx \frac{r_t^f - \delta}{\mu_c}$$

- Typical values for δ in the range from 0.01 to 0.05. For $\delta = 0.02$, one obtains

$$\gamma = \frac{0.01 - 0.02}{0.02} = -0.5.$$

  For $\delta = 0$, one gets $\gamma = 0.5$.
- Moral: To match the risk-free rate one needs an unrealistically low degree of risk-aversion

Asset Pricing Puzzles

- Equity Premium Puzzle / Risk-free Rate Puzzle (Mehra and Prescott 1985; Weil 1993): Simple consumption-based asset pricing model cannot simultaneously explain
  1. high equity premium
  2. low risk-free rates
- Excess Volatility Puzzle (Shiller 1981): Simple consumption-based asset pricing model cannot explain high volatility of stock prices compared to the volatility of consumption/dividend growth.

Way out?

- The model considered so far is too restrictive.
  - Time-additive utility
  - Normally-distributed returns
  - Linear dynamics
- Better specification of preferences
  1. Habit Formation (Abel 1990, Campbell and Cochrane 1999)
  2. Recursive Utility (Epstein and Zin 1989)
- Better consumption / dividend dynamics
  1. Long-run risk model (Bansal and Yaron 2004)
  2. Disaster models (Barro 2006, 2009; Barro and Jin 2015; Gabaix 2008)
- Heterogeneous Agents
- Market Frictions
- Production-based asset pricing
- Partial information
- ...