Agenda

Representative Investor Revisited

- So far, we have assumed the existence of an representative investor.
  - one investor who represents the market
  - equilibrium condition: representative investor has to consume aggregate endowment
- Aggregation might be rather involved.
- This section deals with the question on how to construct a representative investor if agents have
  - heterogenous preferences
  - heterogenous beliefs
  - heterogenous income
  - heterogenous information
  - ...
- Hetergogeneity necessary to generate trading.
- We focus on CRRA utility since the construction of a representative investor with recursive preferences is still an open question.

Framework

- Endowment process = consumption C_t
- Assume that financial markets are complete.
- Assume there is an arbitrary number of individual investors $i = 1, \ldots, n$.
  - endowment: owns fraction $\omega_{i,t}$ of aggregate endowment C_t
  - utility function over consumption:

$$u_i(C_{i,t}) = \frac{1}{1-\gamma_i} C_{i,t}^{1-\gamma_i}$$

  - Time preference rate δ shared by all investors.
- Consumption of each investor in equilibrium?
- Equilibrium asset prices?

Optimization Problem

- Each investor solves the individual optimization problem

$$\max \sum_{t=0}^{T} \mathrm{e}^{-\delta t} \mathbb{E}\big[u_i(C_{i,t})\big]$$

- Euler condition

$$\mathbb{E}_t\left[\mathrm{e}^{-\delta} \frac{u_i'(C_{i,t+1})}{u_i'(C_{i,t})}(1 + r_{t+1})\right] = 1.$$

- Individual pricing kernel is thus

$$M^i_{t,t+1} = \mathrm{e}^{-\delta} \frac{u_i'(C_{i,t+1})}{u_i'(C_{i,t})}.$$

where $M^i_{t,t+1} = M^i_{t+1}/M^i_t$.

Unique Pricing Kernel

- Market completeness implies unique asset prices.
- All investors agree upon the pricing kernel.

$$\mathrm{e}^{-\delta} \frac{u_i'(C_{i,t+1})}{u_i'(C_{i,t})} = \mathrm{e}^{-\delta} \frac{u_j'(C_{j,t+1})}{u_j'(C_{j,t})}$$

- Consequently,

$$\frac{u_i'(C_{i,t})}{u_j'(C_{j,t})} = \frac{u_i'(C_{i,t+1})}{u_j'(C_{j,t+1})}$$

  for all t, i.e., the ratio is constant over time.
- For $t = 0$: set $y_i = u_i'(C_{i,0})$, $y_j = u_j'(C_{j,0})$.

Consumption Sharing Rule

- Market clearing: all investors have to consume the aggregate endowment.

Consumption Sharing Rule

If the investors differ with respect to their utility functions, but have the same subjective time discount rate, then the individual optimal consumption levels at time t are given by

$$\frac{u_i'(C_{i,t})}{u_j'(C_{j,t})} = \frac{y_i}{y_j}, \qquad \sum_{i=1}^{n} C_{i,t} = C_t$$

where $y_i = u_i'(C_{i,0})$, $y_j = u_j'(C_{j,0})$.

- Individual consumption depends on aggregate consumption and on the initial wealth distribution
- It does not depend on the current state of the world.

Construction of the Representative Investor

- Since the market is complete, there is a representative investor.
- Representative investor reflects the whole market.
- Utility of representative investor is a weighted average of individual utility functions

$$U(C_t) = \sum_{i=1}^{n} \lambda_i u_i(C_{i,t})$$

s.t.

$$\sum_{i=1}^{n} C_{i,t} = C_t$$

- Lagrangian

$$\mathcal{L}(C_{1,t}, \dots, C_{n,t}, \alpha) = \sum_{i=1}^{n} \lambda_i u_i(C_{i,t}) - \alpha_t \Big[\sum_{i=1}^{n} C_{i,t} - C_t \Big]$$

Construction of the Representative Investor

- First-order condition

$$\lambda_i u_i'(C_{i,t}) = \alpha_t$$

- Therefore,

$$\lambda_i u_i'(C_{i,t}) = \lambda_j u_j'(C_{j,t})$$

- Dividing by the same condition at time 0 yields the same optimality condition as above

$$\frac{u_i'(C_{i,t})}{u_j'(C_{j,t})} = \frac{u_i'(C_{i,0})}{u_j'(C_{j,0})}$$

- To match them set $\lambda_i = \frac{1}{y_i}$.

Properties of the Representative Investor: Marginal Utility

- We are finally interested in the representative agents degree of risk aversion. Recall (Arrow 1970; Pratt 1966):

$$\mathrm{RRA} = -C\frac{U''(C)}{U'(C)}.$$

- Marginal utility of the representative investor

$$\begin{aligned} U'(C_t) &= \sum_{i=1}^{n} \lambda_i u_i'(C_{i,t})\omega_{i,t} \\ &= \sum_{i=1}^{n} \lambda_1 u_1'(C_{1,t})\omega_{i,t} \\ &= \lambda_1 u_1'(C_{1,t}) \sum_{i=1}^{n} \omega_{i,t} \\ &= \lambda_1 u_1'(C_{1,t}) = \lambda_i u_i'(C_{i,t}) \end{aligned}$$

Properties of the Representative Investor

- Second derivative $U''(C_t) = \lambda_i u_i'(C_{i,t})\omega_{i,t}$. Consequently,

$$\frac{U''(C_t)}{U'(C_t)} = \frac{u_i''(C_{i,t})}{u_i'(C_{i,t})}\omega_{i,t}$$

- Substituting into market clearing condition $\sum_{i=1}^{n} \omega_{i,t} = 1$ yields:

$$\sum_{i=1}^{n} \frac{U''(C_t)}{U'(C_t)} \frac{u_i'(C_{i,t})}{u_i''(C_{i,t})} = 1 \quad \iff \quad \sum_{i=1}^{n} \frac{u_i'(C_{i,t})}{u_i''(C_{i,t})} = \frac{U'(C_t)}{U''(C_t)}$$

- Multiplying by $-\frac{1}{C_t} = -\frac{C_{i,t}}{C_{i,t} C_t}$ yields

$$-\sum_{i=1}^{n} \frac{u_i'(C_{i,t})}{u_i''(C_{i,t}) C_{i,t}} \frac{C_{i,t}}{C_t} = -\frac{U'(C_t)}{U''(C_t) C_t}$$

$$\sum_{i=1}^{n} \frac{1}{\mathrm{RRA_i}} \omega_{i,t} = \frac{1}{\mathrm{RRA}}$$

Example for CRRA Utility: Consumption Sharing

- Investors have CRRA utility with risk aversion γ_i.
- Consumption sharing rule simplifies

$$\frac{C_{i,t}^{-\gamma_i}}{C_{1,t}^{-\gamma_1}} = \frac{y_i}{y_1}, \qquad \sum_{i=1}^{n} C_{i,t} = C_t$$

- Solving the first equation for $C_{i,t}$ and substitute into the market clearing condition

$$\sum_{i=1}^{n} C_{1,t}^{\frac{\gamma_1}{\gamma_i}} \left(\frac{y_i}{y_1}\right)^{-1/\gamma_i} = C_t$$

- can be solved for $C_{1,t}$, which then gives the optimal consumption of all other investors at t

Example for CRRA Utility: Representative Investor

- Optimal consumption at time t
  - increases in aggregate consumption C_t
  - depends on initial wealth distribution (if investor i consumes more than investor j at time 0 he may actually consume less at time t.)
  - depends on risk aversion levels of all investors.
- Representative Investor

$$\sum_{i=1}^{n} \frac{1}{\gamma_i} \frac{C_{i,t}}{C_t} = \frac{1}{\gamma} \quad \Longleftrightarrow \quad \sum_{i=1}^{n} \psi_i \omega_{i,t} = \psi$$

The EIS of the representative investor is thus the weighted average of the EIS of the individual investors. This is not true for recursive utility.

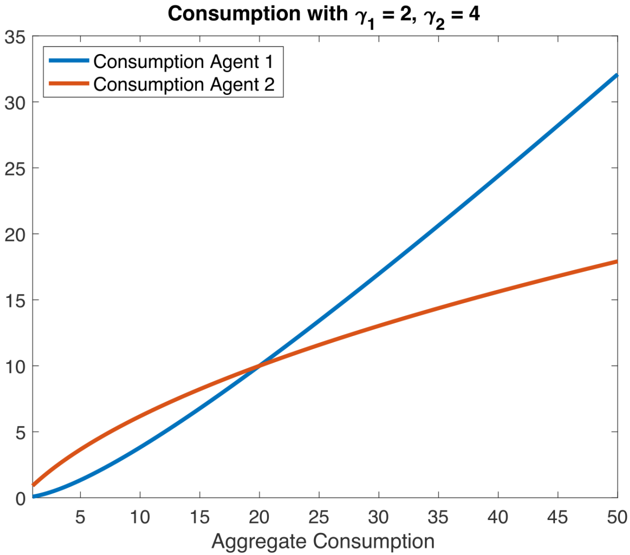
Special Case: Two CRRA Investors

- Special case: two investors, CRRA utility $\gamma_1 < \gamma_2$
- Consumption Sharing Rule

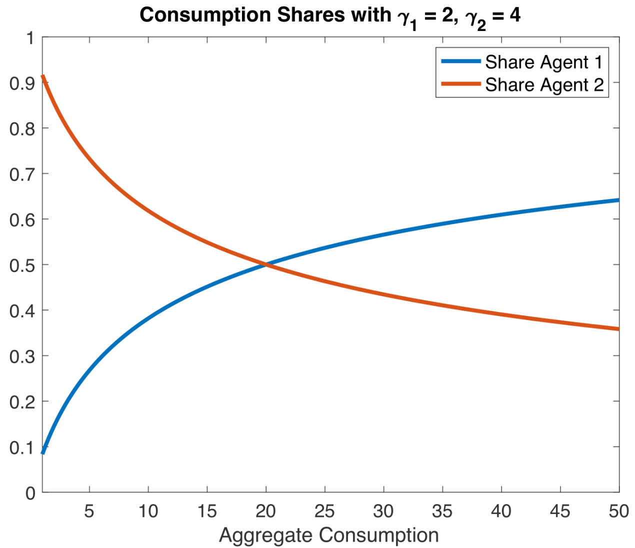
$$\frac{C_{1,t}^{-\gamma_1}}{C_{2,t}^{-\gamma_2}} = \frac{y_1}{y_2}, \qquad C_{1,t} + C_{2,t} = C_t$$

- Less risk averse investor:
  - $C_{1,t}$ is a convex and increasing in C_t
  - consumption share is concave, increases in C_t
  - compensated by more consumption in good states
- More risk averse investor:
  - $C_{2,t}$ is a concave and increasing in C_t
  - consumption share is convex, decreases in C_t
  - willing to give up consumption in good states

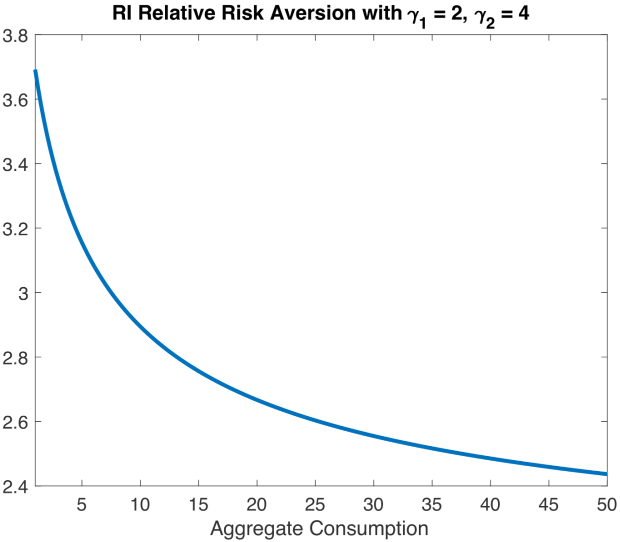
Special Case: Two CRRA Investors



Special Case: Two CRRA Investors



Special Case: Two CRRA Investors



Agenda

Motivation

- Many economic quantities unobservable and hard to assess
- Example: Assume simple i.i.d. consumption dynamics

$$\Delta c_{t+1} = \mu + \sigma \eta_{t+1}$$

- μ not observable $\Rightarrow$ realizations of this model cannot be distinguished with certainty form alternative model with expected growth equal to $\tilde{\mu} \neq \mu$
- Investors can differ w.r.t. their beliefs about μ
  - Investor 1 beliefs $\mu = \mu_1$ and observes

$$\Delta c_{t+1} = \mu_1 + \sigma_1 \eta_{1,t+1}$$

  - Investor 2 beliefs $\mu = \mu_2$ and observes

$$\Delta c_{t+1} = \mu_2 + \sigma_1 \eta_{2,t+1}$$

Framework

- Endowment process = consumption C_t
- Assume that financial markets are complete.
- Assume there is an arbitrary number of individual investors $i = 1, \ldots, n$.
  - endowment: owns fraction $\frac{\partial C_{i,t}}{\partial C_t}$ of aggregate endowment C_t
  - identical utility functions over consumption:
  - Time preference rate δ shared by all investors.
  - subjective beliefs: each investor beliefs in a subjective probability measure $\mathbb{P}^i$.
- Consumption of each investor in equilibrium?
- Equilibrium asset prices?

Optimization Problem

- Each investor solves the individual optimization problem

$$\max \sum_{t=0}^{T} \mathrm{e}^{-\delta t} \mathbb{E}^{\mathbb{P}^i} \left[u(C_{i,t}) \right]$$

- Euler condition

$$\mathbb{E}_t^{\mathbb{P}^i} \left[\mathrm{e}^{-\delta} \frac{u'(C_{i,t+1})}{u'(C_{i,t})} (1 + r_{t+1}) \right] = 1.$$

- Individual pricing kernel is thus

$$M^i_{t,t+1} = \mathrm{e}^{-\delta} \frac{u'(C_{i,t+1})}{u'(C_{i,t})}.$$

where $M^i_{t,t+1} = M^i_{t+1} / M^i_t$.

Unique Pricing Kernel? No...

- ... but all investors agree on the prices of traded assets.

$$\mathbb{E}^{\mathbb{P}^i}[M^i_{0,t}X_t] = \mathbb{E}^{\mathbb{P}^j}[M^j_{0,t}X_t]$$

- Performing a change of measure

$$\mathbb{E}^{\mathbb{P}^i}[M^i_{0,t}X_t] = \mathbb{E}^{\mathbb{P}^i}\left[\frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i} M^j_{0,t}X_t\right].$$

- Market completeness implies that this relation has to hold for all payoffs X_t. Therefore

$$\left.\frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i}\right|_{\mathcal{F}_t} M^j_{0,t} = M^i_{0,t}$$

Consequently,

$$\frac{u'(C_{i,t})}{u'(C_{j,t})} = \frac{u'(C_{i,0})}{u'(C_{j,0})} \left.\frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i}\right|_{\mathcal{F}_t} = \frac{y_i}{y_j} \left.\frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i}\right|_{\mathcal{F}_t}$$

Consumption Sharing Rule

- Now, the results partly carry over (but...)

Consumption Sharing Rule

If the investors differ with respect to their utility functions, but have the same subjective time discount rate, then the individual optimal consumption levels at time t are given by

$$\frac{u'(C_{i,t})}{u'(C_{j,t})} = \frac{y_i}{y_j} \frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i}\Big|_{\mathcal{F}_t}, \qquad \sum_{i=1}^{n} C_{i,t} = C_t$$

where $y_i = u'(C_{i,0}) = y_i$, $y_j = u'(C_{j,0})$.

- Individual consumption depends on aggregate consumption and on the initial wealth distribution
- ... they depend on the disagreement process $\frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i}\Big|_{\mathcal{F}_t}$. Therefore: path-dependent

Example: Two CRRA Investors

- Example: two investors with CRRA utility
- Relation between individual consumption levels

$$\frac{C_{1,t}^{-\gamma}}{C_{2,t}^{-\gamma}} = \frac{C_{1,0}^{-\gamma}}{C_{2,0}^{-\gamma}} \frac{\mathrm{d}\mathbb{P}^1}{\mathrm{d}\mathbb{P}^2}\Big|_{\mathcal{F}_t} \qquad \Rightarrow \qquad C_{2,t} = C_{1,t} \frac{C_{2,0}}{C_{1,0}} \left(\frac{\mathrm{d}\mathbb{P}^1}{\mathrm{d}\mathbb{P}^2}\Big|_{\mathcal{F}_t} \right)^{\frac{1}{\gamma}}$$

- Plugging into market clearing condition:

$$C_{1,t} + C_{1,t} \frac{C_{2,0}}{C_{1,0}} \left(\frac{\mathrm{d}\mathbb{P}^1}{\mathrm{d}\mathbb{P}^2}\Big|_{\mathcal{F}_t} \right)^{\frac{1}{\gamma}} = C_t$$

- Consumption sharing rule:

$$C_{1,t} = \frac{C_t}{1 + \frac{C_{2,0}}{C_{1,0}} \left(\frac{\mathrm{d}\mathbb{P}^1}{\mathrm{d}\mathbb{P}^2}\Big|_{\mathcal{F}_t} \right)^{\frac{1}{\gamma}}}$$

Construction of the Representative Investor

- Since the market is complete, there is a representative investor.
- Representative investor $U(C_t) = \sum_{i=1}^{n} \lambda_i u_i(C_{i,t})$ s.t. $\sum_{i=1}^{n} C_{i,t} = C_t$
- First-order condition

$$\lambda_i u'(C_{i,t}) = \alpha_t$$

- Therefore,

$$\lambda_i u'(C_{i,t}) = \lambda_j u'(C_{j,t})$$

- Coincides with the consumption sharing rule if we set

$$\frac{\lambda_j}{\lambda_i} = \frac{u'(C_{i,0})}{u'(C_{i,0})} \frac{\mathrm{d}\mathbb{P}^j}{\mathrm{d}\mathbb{P}^i}\Big|_{\mathcal{F}_t}$$