Capital Markets and Asset Pricing

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Part III

Option Pricing

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Motivation for State Pricing

- All we have seen so far holds under the assumption that there is no uncertainty.
- The framework from the previous chapter cannot deal with uncertainty about the timing and sizes of the payments.
- Examples:
  - Default of corporate bonds (credit risk)
  - Stock prices (uncertainty about dividend payments)
  - Derivatives
- Thus we need frameworks that can deal with uncertainty.
  - **State Pricing**: Taylor-made for credit risk, but also applicable for stock valuation and option pricing
  - **Binomial Tree / Black-Scholes**: Benchmark models for option pricing
  - **CAPM / APT**: Benchmark models for stock valuation

- Two points in time $t \in \{0, 1\} \implies$ one period
- At $t = 1$ there are S different possible states.
- There are N assets (stocks, bonds) on the market, summarized in a payoff matrix X:

$$X = \begin{array}{cc} \text{Stock} & \text{Bond} \\ \left(\begin{array}{c} 120 \\ 80 \end{array}\right. & \left.\begin{array}{c} 100 \\ 100 \end{array}\right) \end{array} \begin{array}{l} \leftarrow \text{good state} \\ \leftarrow \text{bad state} \end{array}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{S,1} & \cdots & x_{S,N} \end{pmatrix}$$

$x_{s,n}$: Payoff of asset n in state s at $t = 1$

- $S = 2$: one-period binomial model, $S = 3$: one-period trinomial model.
- For illustration purposes, we will only consider $N = 2$, or $N = 3$.

One-period State Pricing Model

- Prices of the assets at $t = 0$ summarized in a price vector

$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}$$

$$p = \begin{pmatrix} 100 \\ 95 \end{pmatrix}$$

- Problem: Find the price p_{N+1} of a new asset (e.g., an option) that is expressed by the following cash-flow vector:

$$\mathrm{CF} = \boxed{\begin{pmatrix} \mathrm{CF}_1 \\ \vdots \\ \mathrm{CF}_S \end{pmatrix}}$$

$$\begin{pmatrix} 100 \\ 95 \end{pmatrix} \quad \begin{matrix} \nearrow \begin{pmatrix} 120 \\ 100 \end{pmatrix} \text{ up} \\ \searrow \begin{pmatrix} 80 \\ 100 \end{pmatrix} \text{ down} \end{matrix}$$

$\boxed{t = 0}$ $\qquad$ $\boxed{t = 1}$

Illustrating Example

- Example for a model with $N = 2$ assets (stock and default-free zero bond) and $S = 2$ states (*up* and *down*):

$C_T = \max\{S_T - K; 0\}$

$p_u = 1 - p_d$

$$\begin{tikzcd} & \begin{pmatrix}120\\100\end{pmatrix} \\ \begin{pmatrix}100\\95\end{pmatrix} \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & \begin{pmatrix}80\\100\end{pmatrix} \end{tikzcd}$$

$95(1+r) = 100$

$\Rightarrow \quad r = 5.26\%$

- Therefore,

$$X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix}, \qquad p = \begin{pmatrix} 100 \\ 95 \end{pmatrix}.$$

- Price of an additional asset with payoff vector $\text{CF} = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$?
- What could this asset represent? $\rightarrow$ Call option with $K=20$

Illustrating Example

$$\begin{tikzcd} & \begin{pmatrix}120\\100\\100\end{pmatrix} \\ \begin{pmatrix}100\\95\\\boxed{p_3}\end{pmatrix} \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\ & \begin{pmatrix}80\\100\\60\end{pmatrix} \end{tikzcd}$$

1. **Replication**
   - Construct a portfolio that replicates the cash flow vector CF of the defaultable bond.
   - According to the Law of One Price, the portfolio and the defaultable bond must have the same price.
2. **State Price Securities**
   - Determine the Price of the so-called Arrow-Debreu securities, which pay $1 if a certain state materializes.
   - Use them to price the derivative.

How can one construct a replication portfolio? $X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix}$

Replication Portfolio

1. The replication portfolio φ solves the linear system $\boxed{\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}}$

$$\begin{cases} \varphi_1 \cdot \boxed{120} + \varphi_2 \cdot \boxed{100} = 100 \\ \varphi_1 \cdot \boxed{80} + \varphi_2 \cdot \boxed{100} = 60 \end{cases}$$

$$X\varphi = \boxed{CF}$$

φ_n denotes the number of assets of type n in the portfolio.

2. The arbitrage-free price of the asset $N+1$ is

$CF = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$ $\quad X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix}$

$$p_{N+1} = p^\top \varphi$$

$$P_3 = 100 \cdot \varphi_1 + 95 \cdot \varphi_2$$

Each row in the linear system represents one state; each column one asset.

Illustrating Example: Replication

1. Replication Portfolio:

$$\begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ -0.2 \end{pmatrix}$$

$120\,\varphi_1 + \boxed{100\,\varphi_2} = 100$ (up) $\quad | \; \mathrm{I} - \mathrm{II}$

$80\,\varphi_1 + \boxed{100\,\varphi_2} = 60$

$40\varphi_1 = 40 \quad |:40 \quad \Rightarrow \boxed{\varphi_1 = 1}$

$\Rightarrow 80 + 100\varphi_2 = 60$

$\Rightarrow \boxed{\varphi_2 = -0.2}$

2. Arbitrage-free price of the derivative

$$\begin{aligned} p_3 &= p_1\varphi_1 + p_2\varphi_2 \\ &= 100 \cdot 1 + 95 \cdot (-0.2) \\ &= 81 \end{aligned}$$

2^{nd} Approach: State Price Securities

What is a state price security?

$$X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix}; \quad X^T = \begin{pmatrix} 120 & 80 \\ 100 & 100 \end{pmatrix}$$

State Price Security

1. An Asset x_s with price $\boxed{\pi_s}$, which pays exactly one dollar in state s and zero else is called a state price security or Arrow-Debreu security.

$$120 \cdot \pi_1 + 80 \pi_2 = 100$$
$$100 \pi_1 + 100 \pi_2 = 95$$

$$X^\top \pi = p \qquad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 95 \end{pmatrix}$$

   π_s denotes the price of the state price security x_s (also known as Arrow-Debreu price).

2. The arbitrage-free price of the asset $N+1$ is

$$p_{N+1} = \mathrm{CF}^\top \pi$$

Each row in the linear system represents one asset; each column one state.

Illustrating Example: State Prices

1. State Price Securities with prices π_u, π_d

$$\begin{pmatrix} 120 & 80 \\ 100 & 100 \end{pmatrix} \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \begin{pmatrix} 100 \\ 95 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \begin{pmatrix} 0.60 \\ 0.35 \end{pmatrix}$$

$120\pi_1 + 80\pi_2 = 100$

$100\pi_1 + 100\pi_2 = 95 \quad | \cdot 1.2$

$\Rightarrow \quad 80\pi_2 - 120\pi_2 = 100 - 95 \cdot 1.2$

$-40\pi_2 = -14 \quad \Rightarrow \quad \pi_2 = \frac{14}{40} = 0.35$

$120\pi_1 + 80 \cdot 0.35 = 100$

$\pi_1 = 0.6$

2. Arbitrage-free price of the derivative

$$\begin{aligned} p_3 &= \boxed{\mathrm{CF}_u}\pi_u + \boxed{\mathrm{CF}_d}\pi_d \\ &= 100 \cdot 0.6 + 60 \cdot 0.35 \\ &= \boxed{81} \end{aligned}$$

Risk-free Asset

$$r = \frac{1}{\pi_1 + \pi_2} - 1 = \frac{1}{0.6 + 0.35} - 1 = 5.26\%$$

- The price of an asset with cash-flow CF_s in state s is given by

Pricing Rule

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \pi_s$$

- The price of the risk-free asset B with payoff 1 in every state is $B_0 = \frac{1}{1+r}$, hence

Risk-free Asset

$$B_0 = \sum_{s=1}^{S} \pi_s, \qquad r = \frac{1}{\sum_{s=1}^{S} \pi_s} - 1$$

From State Pricing to Risk-neutral Pricing

- Does this procedure always lead to arbitrage-free prices?

No-arbitrage Condition

The market is free of arbitrage if and only if $\pi_s > 0$ for all states.

- Is replication always possible?

Completeness

Replication works if the market is complete. Rule of thumb: if the number of (independent) assets equals the number of states, then the market is complete and every security can be replicated.

- Example for an incomplete market?

$$\begin{pmatrix}100\\95\end{pmatrix} \quad \begin{matrix} \nearrow \begin{pmatrix}120\\100\end{pmatrix} \\ \rightarrow \begin{pmatrix}100\\100\end{pmatrix} \\ \searrow \begin{pmatrix}80\\100\end{pmatrix} \end{matrix} \quad \begin{pmatrix}100\\80\\60\end{pmatrix}$$

incomplete since there are more states than assets

From State Pricing to Risk-neutral Pricing

- Have the state price securities something to do with probabilities?
- Yes! But not with the real probabilities ...
- Define

$$q_s = \pi_s(1+r) \neq \underbrace{p_s}_{\text{real-world prob.}}$$

Risk-neutral Probabilities

If the market is free of arbitrage,

$$q_s > 0, \quad \sum_{s=1}^{S} q_s = 1$$

form a set of probabilites, the so-called risk-neutral probabilities.

$q_u = \pi_u(1+r) = 0.6 \cdot 1.0526 = 63.16\%$

$q_d = 1 - 63.16\% = 36.84\%$

From State Pricing to Risk-neutral Pricing

- If the market is free of arbitrage, prices can be expressed as

$\pi_s = \frac{q_s}{1+r}$

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \boxed{\pi_s} = \sum_{s=1}^{S} \mathrm{CF}_s \boxed{\frac{q_s}{1+r}}$$

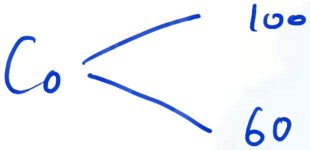
- Consequently, prices can be expressed as discounted expected cash-flows!

Risk-neutral Pricing

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \frac{q_s}{1+r} = \boxed{\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[\mathrm{CF}]}$$

Prices are discounted expected cash-flows

- This is a fundamental insight that holds in much more general markets!
- **1st Warning:** The risk-neutral probabilities are different from the real physical probabilities.
- **2nd Warning:** Investors are not risk-neutral, but prices are formed as if they were risk-neutral (but under **different probabilities**).



$K = 20$

Call

$$C_0 = \mathbb{E}^{\mathbb{Q}}\left[\frac{CF}{1+r}\right] = \underbrace{0.6316}_{q_u} \cdot \frac{100}{1.0526} + \underbrace{0.3684} \cdot \frac{60}{1.0526}$$

$$= 81$$

Summary: State Pricing in a Complete Market

- State prices π_s are the prices for a security paying one dollar in state s and zero else.
- They can be determined by solving the linear system $X^\top \pi = p$, which has a unique solution if and only if the market is complete.
- The market is free of arbitrage if and only if $\pi_s > 0$ for all states s. If the market is arbitrage-free, the risk-neutral probabilities exists and are compounded state prices, i.e., $q_s = \pi_s(1+r)$.
- Given a vector of state prices π or risk-neutral probabilities q, the price of an asset with cash-flow vector CF is given by

$$P_0 = \boxed{\sum_{s=1}^{S} \mathrm{CF}_s \pi_s} = \boxed{\sum_{s=1}^{S} \mathrm{CF}_s \frac{q_s}{1+r}.}$$

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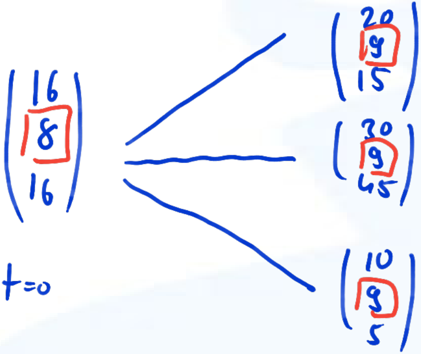
Example: Three States

Assets \ States	1	2	3
1	20	30	10
2	9	9	9
3	15	45	5

$$p = \begin{pmatrix} 16 \\ 8 \\ 16 \end{pmatrix}$$



Example: Three States

1st asset $\quad 20\pi_1 + 30\pi_2 + 10\pi_3 = 16$

2nd asset $\quad 9\pi_1 + 9\pi_2 + 9\pi_3 = 8$

3rd asset $\quad 15\pi_1 + 45\pi_2 + 5\pi_3 = 16$

$\Rightarrow$ Gauss algorithm delivers

$\pi_1 = 0.2667$

$\pi_2 = 0.2222$

$\pi_3 = 0.4$

$$\Rightarrow \quad r = \frac{1}{\sum \pi_s} - 1 = \frac{1}{0.2667 + 0.2222 + 0.4} - 1$$

$$= 12.5\%$$

Example: Three States

Risk-neutral probabilities

$$q_1 = \pi_1 \cdot (1+r) = 0.2667 \cdot 1{,}125 = 0.3$$
$$q_2 = = 0{,}2222 \cdot 1.125 = 0{,}25$$
$$q_3 = = 0.4 \cdot 1.125 = \underline{0.45}$$
$$1{,}00$$

- complete market due to the unique solution
- arbitrage-free market since $\pi_s > 0$ for all states

Now, we want to price a put option on asset 1 with $K = 15$:

Example: Three States

	S_1	P
S_0, P_0	20	0
	30	0
	10	5

$$P_T = \max\{\underset{15-20}{K - S_T}\,;\,0\}$$

(Put option, see end of chapter 2)

$$P_0 = \frac{1}{1+r}\left(q_1 \cdot P_{T,1} + q_2 \cdot P_{T,2} + q_3 \cdot P_{T,3}\right)$$
$$= \frac{1}{1.125}\left(0.3 \cdot 0 + 0.25 \cdot 0 + 0.45 \cdot 5\right)$$
$$= 2$$

$$P_0 = C_0 - S_0 + \frac{K}{1+r} \Rightarrow C_0 = P_0 + S_0 - \frac{K}{1+r}$$
$$= 2 + 16 - \frac{15}{1.125} = 4.67$$

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Structure of One Period

$$
\begin{tikzcd}
 & \begin{pmatrix} S_0 \cdot (1+u) \\ B_0 \cdot (1+r) \end{pmatrix} \\
\begin{pmatrix} S_0 \\ B_0 \end{pmatrix} \arrow[ur, "p_u"] \arrow[dr, "p_d"'] & \\
 & \begin{pmatrix} S_0 \cdot (1+d) \\ B_0 \cdot (1+r) \end{pmatrix}
\end{tikzcd}
$$

- Set $B_0 = 1$. Then:

$$X = \begin{pmatrix} S_0 \cdot (1+u) & 1+r \\ S_0 \cdot (1+d) & 1+r \end{pmatrix}, \qquad p = \begin{pmatrix} S_0 \\ 1 \end{pmatrix}, \qquad u > r > d$$

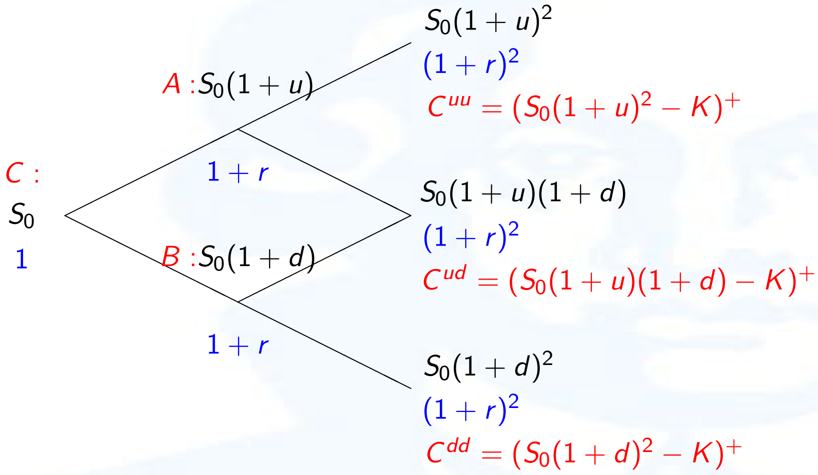
- Determine the price C_0 of a derivative with payoff $C = \begin{pmatrix} C^u \\ C^d \end{pmatrix}$.
- One obtains

$$C_0 = \frac{1}{1+r} \cdot \left[C^u \underbrace{\frac{r-d}{u-d}}_{=q_u} + C^d \underbrace{\frac{u-r}{u-d}}_{=q_d} \right]$$

Binomial Trees

- If we want to price financial derivatives, the one-period state pricing model is too simplistic.
- We extend the idea from the one-period model to a binomial tree.
- We consider trees with one stock S and one risk-free asset B.
- The risk-free rate is exogeneously given and denoted by r.
- In each period, the stock can either increase by u or decrease by d.
- We assume $u > r > d$. This condition ensures that the market is free-of arbitrage.
- The risk-neutral probability for an up-state is given by $q = q_u = \frac{r-d}{u-d}$.

Two-Period Model



- Node A:

$$C^u = \frac{1}{1+r}\left[qC^{uu} + (1-q)C^{ud}\right]$$

- Node B:

$$C^d = \frac{1}{1+r}\left[qC^{ud} + (1-q)C^{dd}\right]$$

- Node C:

$$\begin{aligned} C_0 &= \frac{1}{1+r}\left[qC^u + (1-q)C^d\right] \\ \Rightarrow \boxed{C_0 = \frac{1}{(1+r)^2}\left[q^2C^{uu} + 2q(1-q)C^{ud} + (1-q)^2C^{dd}\right]} \end{aligned}$$

Multi-Period Model: Binomial Coefficients

Multi-Period Model

- Extending this idea to an arbitrary number of periods leads to the following closed-form solution

$$C_0 = \frac{1}{(1+r)^T} \sum_{i=0}^{T} \binom{T}{i} q^i (1-q)^{T-i} C_T^{(i)}$$

where $\binom{T}{i} = \frac{T!}{i!(T-i)!}$ denotes the binomial coefficient. It counts the number of paths leading to node i. $\rightarrow$ Pascal's triangle

- For a call option the terminal payoff is given by

$$C_T^{(i)} = \left(S_0 \left(1+u\right)^i \left(1+d\right)^{T-i} - K\right)^+$$

- For a put option the terminal payoff is given by

$$C_T^{(i)} = \left(K - S_0 \left(1+u\right)^i \left(1+d\right)^{T-i}\right)^+$$

Example: Multi-Period Model

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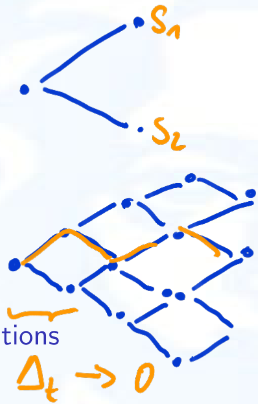


Black-Scholes-Model

- Mathematically involved model in continuous time; stock price:

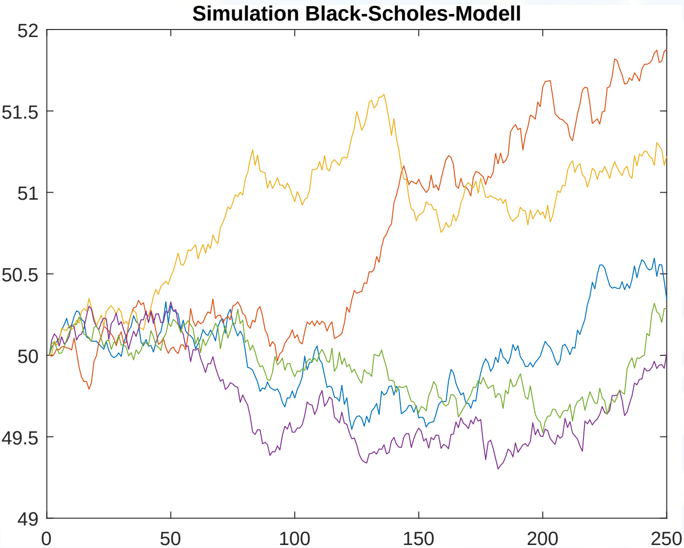
$$\mathrm{d}S_t = S_t\mu\mathrm{d}t + S_t\sigma\mathrm{d}W_t$$

- W_t Wiener process; normally distributed returns
- Interpretation for "small" time step Δt:

$$\boxed{S_{t+\Delta t}} - \boxed{S_t} = S_t\mu\Delta t + S_t\sigma\underbrace{\left(W_{t+\Delta t} - W_t\right)}_{\sim\mathcal{N}(0,\Delta t)}$$

- Idea: Start with a binomial model and increase the number of time steps in $[0, T]$.
- In the limit the binomial model converges to the continuous Black-Scholes model.

Simulation of the Black-Scholes Model



Black-Scholes Formula

- A call (put) option with maturity at time T has the terminal payoff

$$C_T = \max\{S_T - K; 0\}, \qquad P_T = \max\{K - S_T; 0\}$$

- In the more complicated Black-Scholes setting, its current price is given by

$$C_0 = \mathbb{E}^{\mathbb{Q}}[C_T]e^{-rT}$$

- After some painful calculations, one obtains

Black-Scholes Formula

$$C_0 = S_0 \underbrace{\Phi(d_1)}_{\text{CDF of } \mathcal{N}(0,1)} - K \cdot \mathrm{e}^{-rT}\Phi(d_2), \qquad P_0 = C_0 - S_0 + K \cdot \mathrm{e}^{-rT}$$

with

$$\boxed{d_1 = \frac{\ln(S_0/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}},} \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Example: Black-Scholes Formula

$S_0 = 95, \quad K = 100, \quad r = 1\%, \quad T = 2, \quad \sigma = 20\%.$

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(95/100) + (0.01 + \frac{1}{2}\cdot 0.2^2)\cdot 2}{0.2\sqrt{2}}$$

$$= 0.0308$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.0308 - 0.2\sqrt{2} = -0.252$$

$$\Phi(d_1) = 0.5123, \quad \Phi(d_2) = 0.4005$$

$$C_0 = S\cdot\Phi(d_1) - Ke^{-r\cdot T}\Phi(d_2)$$

$$= 95\cdot 0.5123 - 100\cdot e^{-0.01\cdot 2}\cdot 0.4005$$

$$= 9.41$$

Application: Convertible Bond

- Structure of a convertible bond
  - Fixed coupon payments at a rate c until maturity.
  - At maturity, the *buyer* of the convertible bond has the right (but not the obligation) to reclaim the notional N or to claim a number k of stocks of the emitting company, i.e., $k \cdot S_T$.
- The buyer will claim the stocks if the stocks are worth more than the notional, $k \cdot S_T > N$, i.e., if $k \cdot S_T - N > 0$.
  $\Rightarrow$ Buyer holds a call option!
- Payoff structure:

time	1	2	...	T
Cash flow	c	c	...	$c + N + \max\{k \cdot S_T - N, 0\}$

  with $k = N/S_0$
- Structure: guaranteed coupon payments + long call option.

Application: Reverse Convertible Bond

- Structure of a convertible bond
  - Fixed coupon payments at a rate c until maturity.
  - At maturity, the *emitter* of the reverse convertible bond has the right (but not the obligation) to pay back the notional N or to deliver a number k of stocks, i.e., $k \cdot S_T$.
- The emitter will deliver the stocks if the stocks are worth less than the notional, $k \cdot S_T < N$, i.e., if $N - k \cdot S_T > 0$.
  $\Rightarrow$ Buyer is short in a put option!
- Payoff structure:

time	1	2	...	T
Cash flow	c	c	...	$c + N - \max\{N - k \cdot S_T, 0\}$

  with $k = N/S_0$
- Structure: guaranteed coupon payments – put option.

Critique: Black-Scholes Model

σ

- Volatility, interest rate, expected return are assumed to be constant.
  $\longrightarrow$ Volatility Smile
- Returns are assumed to be normally distributed. $\longrightarrow$ Underestimation of extreme events.
- Model builds upon a complete market without frictions (no taxes, transaction costs, short-selling constraints, ...).
- Implied volatility $\neq$ historical volatility
  - These caveats become visible if one investigates what volatilities are necessary to explain option prices by the Black-Scholes formula.
  - Implied volatility is not constant, but depends on K and T.
  - If the option is at-the-money, implied volatility is lowest (volatility smile).

Volatility Smile

$$\underbrace{C_0} = S_0\,\Phi(d_1(\hat{\sigma})) - K e^{-rT}\,\Phi(d_2(\hat{\sigma}))$$