

Capital Markets and Asset Pricing

Dr. Christoph Hambel

Goethe University Frankfurt & Goethe Business School
Faculty of Business and Economics
Department of Finance

Summer Term 2022

Part III

Option Pricing

Table of Contents

- 1 State Pricing in a Nutshell
- 2 An Example with Three States
- 3 Binomial Trees
- 4 Black-Scholes Model and Applications

Motivation for State Pricing

- All we have seen so far holds under the assumption that there is no uncertainty.
- The framework from the previous chapter cannot deal with uncertainty about the timing and sizes of the payments.
- Examples:
 - Default of corporate bonds (credit risk)
 - Stock prices (uncertainty about dividend payments)
 - Derivatives
- Thus we need frameworks that can deal with uncertainty.
 - **State Pricing:** Taylor-made for credit risk, but also applicable for stock valuation and option pricing
 - **Binomial Tree / Black-Scholes:** Benchmark models for option pricing
 - **CAPM / APT:** Benchmark models for stock valuation

One-period State Pricing Model

- Two points in time $t \in \{0, 1\} \implies$ one period
- At $t = 1$ there are S different possible states.
- There are N assets (stocks, bonds) on the market, summarized in a payoff matrix X :

$$X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix} \begin{matrix} \leftarrow \text{Good stock} \\ \leftarrow \text{Bad stock} \end{matrix} \quad X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{S,1} & \cdots & x_{S,N} \end{pmatrix} \begin{matrix} \leftarrow \text{Good stock} \\ \leftarrow \text{Bad stock} \end{matrix}$$

Handwritten notes: 'Stock' and 'Bond' with arrows pointing to the columns of the matrix. 'Good stock' and 'Bad stock' with arrows pointing to the rows of the matrix.

$x_{s,n}$: Payoff of asset n in state s at $t = 1$

- $S = 2$: one-period binomial model, $S = 3$: one-period trinomial model.
- For illustration purposes, we will only consider $N = 2$, or $N = 3$.

One-period State Pricing Model

- Prices of the assets at $t = 0$ summarized in a price vector

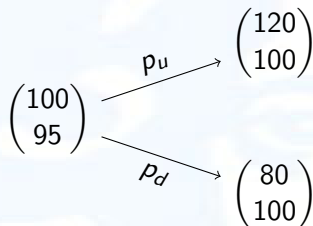
$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} \quad p = \begin{pmatrix} 100 \\ 95 \end{pmatrix}$$

- Problem: Find the price p_{N+1} of a new asset (e.g., an option) that is expressed by the following cash-flow vector:

$$CF = \begin{pmatrix} CF_1 \\ \vdots \\ CF_S \end{pmatrix} \quad \begin{pmatrix} 100 \\ 99 \end{pmatrix} \begin{matrix} \nearrow \begin{pmatrix} 120 \\ 100 \end{pmatrix} \\ \searrow \begin{pmatrix} 80 \\ 100 \end{pmatrix} \end{matrix}$$

Illustrating Example

- Example for a model with $N = 2$ assets (stock and default-free zero bond) and $S = 2$ states (*up* and *down*):

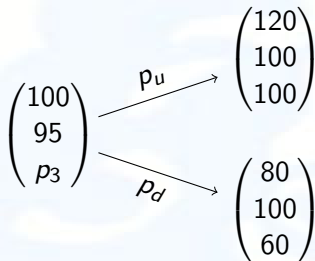


- Therefore,

$$X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix}, \quad p = \begin{pmatrix} 100 \\ 95 \end{pmatrix}.$$

- Price of an additional asset with payoff vector $CF = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$?
- What could this asset represent? \rightarrow

Illustrating Example



1 Replication

- Construct a portfolio that replicates the cash flow vector CF of the defaultable bond.
- According to the Law of One Price, the portfolio and the defaultable bond must have the same price.

2 State Price Securities

- Determine the Price of the so-called Arrow-Debreu securities, which pay \$1 if a certain state materializes.
- Use them to price the derivative.

1st Approach: Replication

How can one construct a replication portfolio?

Replication Portfolio

- 1 The replication portfolio φ solves the linear system

$$X\varphi = CF$$

φ_n denotes the number of assets of type n in the portfolio.

- 2 The arbitrage-free price of the asset $N + 1$ is

$$p_{N+1} = p^\top \varphi$$

Each row in the linear system represents one state; each column one asset.

Illustrating Example: Replication

1 Replication Portfolio:

$$\begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ -0.2 \end{pmatrix}$$

2 Arbitrage-free price of the derivative

$$\begin{aligned} p_3 &= p_1\varphi_1 + p_2\varphi_2 \\ &= 100 \cdot 1 + 95 \cdot (-0.2) \\ &= 81 \end{aligned}$$

What is a state price security?

State Price Security

- 1 An Asset x_s with price π_s , which pays exactly one dollar in state s and zero else is called a state price security or Arrow-Debreu security.

$$X^T \pi = p$$

π_s denotes the price of the state price security x_s (also known as Arrow-Debreu price).

- 2 The arbitrage-free price of the asset $N + 1$ is

$$p_{N+1} = CF^T \pi$$

Each row in the linear system represents one asset; each column one state.

Illustrating Example: State Prices

- ① State Price Securities with prices π_u, π_d

$$\begin{pmatrix} 120 & 80 \\ 100 & 100 \end{pmatrix} \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \begin{pmatrix} 100 \\ 95 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \begin{pmatrix} 0.60 \\ 0.35 \end{pmatrix}$$

- ② Arbitrage-free price of the derivative

$$\begin{aligned} p_3 &= CF_u \pi_u + CF_d \pi_d \\ &= 100 \cdot 0.6 + 60 \cdot 0.35 \\ &= 81 \end{aligned}$$

Risk-free Asset

- The price of an asset with cash-flow CF_s in state s is given by

Pricing Rule

$$P_0 = \sum_{s=1}^S CF_s \pi_s$$

- The price of the risk-free asset B with payoff 1 in every state is $B_0 = \frac{1}{1+r}$, hence

Risk-free Asset

$$B_0 = \sum_{s=1}^S \pi_s, \quad r = \frac{1}{\sum_{s=1}^S \pi_s} - 1$$

From State Pricing to Risk-neutral Pricing

- Does this procedure always lead to arbitrage-free prices?

No-arbitrage Condition

The market is free of arbitrage if and only if $\pi_s > 0$ for all states.

- Is replication always possible?

Completeness

Replication works if the market is complete. Rule of thumb: if the number of (independent) assets equals the number of states, then the market is complete and every security can be replicated.

- Example for an incomplete market?

From State Pricing to Risk-neutral Pricing

- Have the state price securities something to do with probabilities?
- Yes! But not with the real probabilities ...
- Define

$$q_s = \pi_s(1 + r)$$

Risk-neutral Probabilities

If the market is free of arbitrage,

$$q_s > 0, \quad \sum_{s=1}^S q_s = 1$$

form a set of probabilities, the so-called risk-neutral probabilities.

From State Pricing to Risk-neutral Pricing

- If the market is free of arbitrage, prices can be expressed as

$$P_0 = \sum_{s=1}^S CF_s \pi_s = \sum_{s=1}^S CF_s \frac{q_s}{1+r}$$

- Consequently, prices can be expressed as discounted expected cash-flows!

Risk-neutral Pricing

$$P_0 = \sum_{s=1}^S CF_s \frac{q_s}{1+r} = \frac{1}{1+r} \mathbb{E}^Q[CF]$$

- This is a fundamental insight that holds in much more general markets!
- **1st Warning:** The risk-neutral probabilities are different from the real physical probabilities.
- **2nd Warning:** Investors are not risk-neutral, but prices are formed as if they were risk-neutral (but under **different probabilities**).

Example: Risk-neutral Pricing



Summary: State Pricing in a Complete Market

- State prices π_s are the prices for a security paying one dollar in state s and zero else.
- They can be determined by solving the linear system $X^\top \pi = p$, which has a unique solution if and only if the market is complete.
- The market is free of arbitrage if and only if $\pi_s > 0$ for all states s . If the market is arbitrage-free, the risk-neutral probabilities exist and are compounded state prices, i.e., $q_s = \pi_s(1 + r)$.
- Given a vector of state prices π or risk-neutral probabilities q , the price of an asset with cash-flow vector CF is given by

$$P_0 = \sum_{s=1}^S CF_s \pi_s = \sum_{s=1}^S CF_s \frac{q_s}{1 + r}.$$

Table of Contents

- 1 State Pricing in a Nutshell
- 2 An Example with Three States**
- 3 Binomial Trees
- 4 Black-Scholes Model and Applications

Example: Three States



Example: Three States



Example: Three States



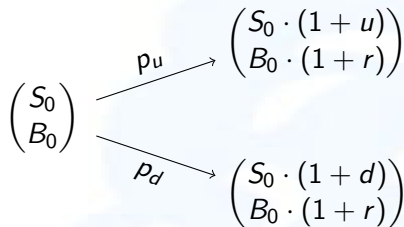
Example: Three States



Table of Contents

- 1 State Pricing in a Nutshell
- 2 An Example with Three States
- 3 Binomial Trees
- 4 Black-Scholes Model and Applications

Structure of One Period



- Set $B_0 = 1$. Then:

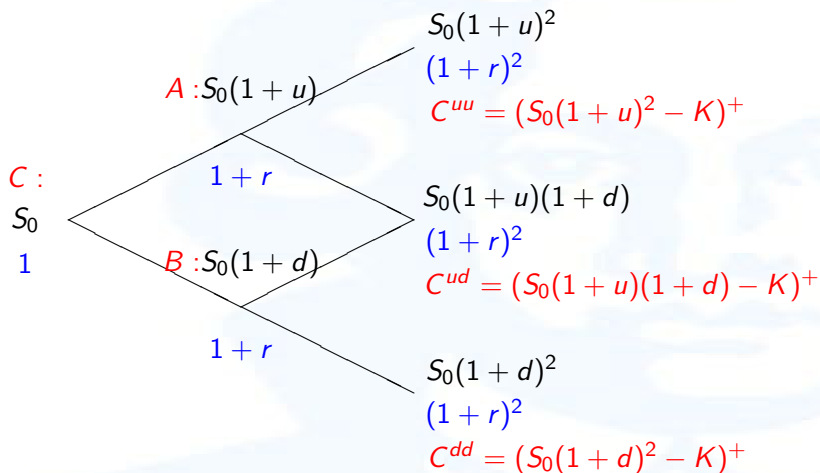
$$X = \begin{pmatrix} S_0 \cdot (1+u) & 1+r \\ S_0 \cdot (1+d) & 1+r \end{pmatrix}, \quad p = \begin{pmatrix} S_0 \\ 1 \end{pmatrix}, \quad u > r > d$$

- Determine the price C_0 of a derivative with payoff $C = \begin{pmatrix} C^u \\ C^d \end{pmatrix}$.
- One obtains

$$C_0 = \frac{1}{1+r} \cdot \left[C^u \underbrace{\frac{r-d}{u-d}}_{=q_u} + C^d \underbrace{\frac{u-r}{u-d}}_{=q_d} \right]$$

- If we want to price financial derivatives, the one-period state pricing model is too simplistic.
- We extend the idea from the one-period model to a binomial tree.
- We consider trees with one stock S and one risk-free asset B .
- The risk-free rate is exogeneously given and denoted by r .
- In each period, the stock can either increase by u or decrease by d .
- We assume $u > r > d$. This condition ensures that the market is free-of arbitrage.
- The risk-neutral probability for an up-state is given by $q = q_u = \frac{r-d}{u-d}$.

Two-Period Model



Two-Period Model

- Node A:

$$C^u = \frac{1}{1+r} \left[qC^{uu} + (1-q)C^{ud} \right]$$

- Node B:

$$C^d = \frac{1}{1+r} \left[qC^{ud} + (1-q)C^{dd} \right]$$

- Node C:

$$C_0 = \frac{1}{1+r} \left[qC^u + (1-q)C^d \right]$$
$$\Rightarrow C_0 = \frac{1}{(1+r)^2} \left[q^2C^{uu} + 2q(1-q)C^{ud} + (1-q)^2C^{dd} \right]$$

Multi-Period Model: Binomial Coefficients



Multi-Period Model

- Extending this idea to an arbitrary number of periods leads to the following closed-form solution

$$C_0 = \frac{1}{(1+r)^T} \sum_{i=0}^T \binom{T}{i} q^i (1-q)^{T-i} C_T^{(i)}$$

where $\binom{T}{i} = \frac{T!}{i!(T-i)!}$ denotes the binomial coefficient. It counts the number of paths leading to node i . \rightarrow Pascal's triangle

- For a call option the terminal payoff is given by

$$C_T^{(i)} = (S_0 (1+u)^i (1+d)^{T-i} - K)^+$$

- For a put option the terminal payoff is given by

$$C_T^{(i)} = (K - S_0 (1+u)^i (1+d)^{T-i})^+$$

Example: Multi-Period Model



Table of Contents

- 1 State Pricing in a Nutshell
- 2 An Example with Three States
- 3 Binomial Trees
- 4 Black-Scholes Model and Applications

- Mathematically involved model in continuous time; stock price:

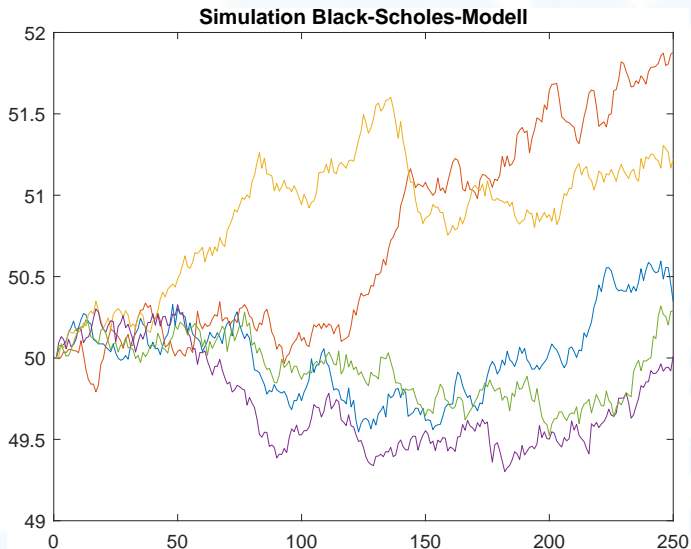
$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

- W_t Wiener process; normally distributed returns
- Interpretation for “small” time step Δt :

$$S_{t+\Delta t} - S_t = S_t \mu \Delta t + S_t \sigma \underbrace{(W_{t+\Delta t} - W_t)}_{\sim \mathcal{N}(0, \Delta t)}$$

- Idea: Start with a binomial model and increase the number of time steps in $[0, T]$.
- In the limit the binomial model converges to the continuous Black-Scholes model.

Simulation of the Black-Scholes Model



Black-Scholes Formula

- A call (put) option with maturity at time T has the terminal payoff

$$C_T = \max\{S_T - K; 0\}, \quad P_T = \max\{K - S_T; 0\}$$

- In the more complicated Black-Scholes setting, its current price is given by

$$C_0 = \mathbb{E}^{\mathbb{Q}}[C_T]e^{-rT}$$

- After some painful calculations, one obtains

Black-Scholes Formula

$$C_0 = S_0\Phi(d_1) - K \cdot e^{-rT}\Phi(d_2), \quad P_0 = C_0 - S_0 + K \cdot e^{-rT}$$

with

$$d_1 = \frac{\ln(S_0/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Example: Black-Scholes Formula



Application: Convertible Bond

- Structure of a convertible bond
 - Fixed coupon payments at a rate c until maturity.
 - At maturity, the *buyer* of the convertible bond has the right (but not the obligation) to reclaim the notional N or to claim a number k of stocks of the emitting company, i.e., $k \cdot S_T$.
- The buyer will claim the stocks if the stocks are worth more than the notional, $k \cdot S_T > N$, i.e., if $k \cdot S_T - N > 0$.
 \Rightarrow Buyer holds a call option!
- Payoff structure:

time	1	2	...	T
Cash flow	c	c	...	$c + N + \max\{k \cdot S_T - N, 0\}$

with $k = N/S_0$

- Structure: guaranteed coupon payments + long call option.

Application: Reverse Convertible Bond

- Structure of a convertible bond
 - Fixed coupon payments at a rate c until maturity.
 - At maturity, the *emitter* of the reverse convertible bond has the right (but not the obligation) to pay back the notional N or to deliver a number k of stocks, i.e., $k \cdot S_T$.
- The emitter will deliver the stocks if the stocks are worth less than the notional, $k \cdot S_T < N$, i.e., if $N - k \cdot S_T > 0$.
 \Rightarrow Buyer is short in a put option!
- Payoff structure:

time	1	2	...	T
Cash flow	c	c	...	$c + N - \max\{N - k \cdot S_T, 0\}$

with $k = N/S_0$

- Structure: guaranteed coupon payments – put option.

Critique: Black-Scholes Model

- Volatility, interest rate, expected return are assumed to be constant.
→ Volatility Smile
- Returns are assumed to be normally distributed. → Underestimation of extreme events.
- Model builds upon a complete market without frictions (no taxes, transaction costs, short-selling constraints, ...).
- Implied volatility \neq historical volatility
 - These caveats become visible if one investigates what volatilities are necessary to explain option prices by the Black-Scholes formula.
 - Implied volatility is not constant, but depends on K and T .
 - If the option is at-the-money, implied volatility is lowest (volatility smile).

Volatility Smile

