# Capital Markets and Asset Pricing

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# Part III

# Option Pricing

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# Motivation for State Pricing

- All we have seen so far holds under the assumption that there is no uncertainty.
- The framework from the previous chapter cannot deal with uncertainty about the timing and sizes of the payments.
- Examples:
  - Default of corporate bonds (credit risk)
  - Stock prices (uncertainty about dividend payments)
  - Derivatives
- Thus we need frameworks that can deal with uncertainty.
  - **State Pricing**: Taylor-made for credit risk, but also applicable for stock valuation and option pricing
  - **Binomial Tree / Black-Scholes**: Benchmark models for option pricing
  - **CAPM / APT**: Benchmark models for stock valuation

# One-period State Pricing Model

- Two points in time $t \in \{0, 1\} \implies$ one period
- At $t = 1$ there are $S$ different possible states.
- There are $N$ assets (stocks, bonds) on the market, summarized in a payoff matrix $X$:

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,N} \\ \vdots & \ddots & \vdots \\ x_{S,1} & \cdots & x_{S,N} \end{pmatrix}$$

  $x_{s,n}$: Payoff of asset $n$ in state $s$ at $t = 1$
- $S = 2$: one-period binomial model, $S = 3$: one-period trinomial model.
- For illustration purposes, we will only consider $N = 2$, or $N = 3$.

# One-period State Pricing Model

- Prices of the assets at $t = 0$ summarized in a price vector

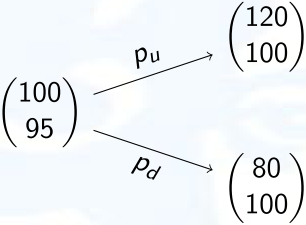
$$p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}$$

- Problem: Find the price $p_{N+1}$ of a new asset (e.g., an option) that is expressed by the following cash-flow vector:

$$\mathrm{CF} = \begin{pmatrix} \mathrm{CF}_1 \\ \vdots \\ \mathrm{CF}_S \end{pmatrix}$$

# Illustrating Example

- Example for a model with $N = 2$ assets (stock and default-free zero bond) and $S = 2$ states (*up* and *down*):

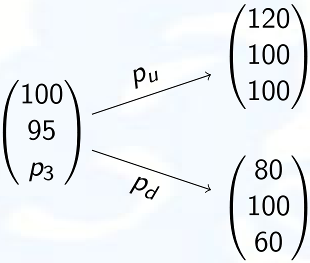



- Therefore,

$$X = \begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix}, \qquad p = \begin{pmatrix} 100 \\ 95 \end{pmatrix}.$$

- Price of an additional asset with payoff vector $\text{CF} = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$?
- What could this asset represent? $\rightarrow$

# Illustrating Example




1. **Replication**
   - Construct a portfolio that replicates the cash flow vector CF of the defaultable bond.
   - According to the Law of One Price, the portfolio and the defaultable bond must have the same price.
2. **State Price Securities**
   - Determine the Price of the so-called Arrow-Debreu securities, which pay $1 if a certain state materializes.
   - Use them to price the derivative.

# $1^{st}$ Approach: Replication

How can one construct a replication portfolio?

## Replication Portfolio

1. The replication portfolio $\varphi$ solves the linear system

$$X\varphi = \text{CF}$$

   $\varphi_n$ denotes the number of assets of type $n$ in the portfolio.

2. The arbitrage-free price of the asset $N+1$ is

$$p_{N+1} = p^\top \varphi$$

Each row in the linear system represents one state; each column one asset.

# Illustrating Example: Replication

1. Replication Portfolio:

$$\begin{pmatrix} 120 & 100 \\ 80 & 100 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 100 \\ 60 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 1.0 \\ -0.2 \end{pmatrix}$$

2. Arbitrage-free price of the derivative

$$\begin{aligned} p_3 &= p_1\varphi_1 + p_2\varphi_2 \\ &= 100 \cdot 1 + 95 \cdot (-0.2) \\ &= 81 \end{aligned}$$

# $2^{\text{nd}}$ Approach: State Price Securities

What is a state price security?

## State Price Security

1. An Asset $x_s$ with price $\pi_s$, which pays exactly one dollar in state $s$ and zero else is called a state price security or Arrow-Debreu security.

$$X^\top \pi = p$$

   $\pi_s$ denotes the price of the state price security $x_s$ (also known as Arrow-Debreu price).

2. The arbitrage-free price of the asset $N + 1$ is

$$p_{N+1} = \mathrm{CF}^\top \pi$$

Each row in the linear system represents one asset; each column one state.

# Illustrating Example: State Prices

1. State Price Securities with prices $\pi_u, \pi_d$

$$\begin{pmatrix} 120 & 80 \\ 100 & 100 \end{pmatrix} \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \begin{pmatrix} 100 \\ 95 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \pi_u \\ \pi_d \end{pmatrix} = \begin{pmatrix} 0.60 \\ 0.35 \end{pmatrix}$$

2. Arbitrage-free price of the derivative

$$\begin{aligned} p_3 &= \text{CF}_u \pi_u + \text{CF}_d \pi_d \\ &= 100 \cdot 0.6 + 60 \cdot 0.35 \\ &= 81 \end{aligned}$$

# Risk-free Asset

- The price of an asset with cash-flow $\mathrm{CF}_s$ in state $s$ is given by

**Pricing Rule**

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \pi_s$$

- The price of the risk-free asset $B$ with payoff 1 in every state is $B_0 = \frac{1}{1+r}$, hence

**Risk-free Asset**

$$B_0 = \sum_{s=1}^{S} \pi_s, \qquad r = \frac{1}{\sum_{s=1}^{S} \pi_s} - 1$$

# From State Pricing to Risk-neutral Pricing

- Does this procedure always lead to arbitrage-free prices?

## No-arbitrage Condition

The market is free of arbitrage if and only if $\pi_s > 0$ for all states.

- Is replication always possible?

## Completeness

Replication works if the market is complete. Rule of thumb: if the number of (independent) assets equals the number of states, then the market is complete and every security can be replicated.

- Example for an incomplete market?

# From State Pricing to Risk-neutral Pricing

- Have the state price securities something to do with probabilities?
- Yes! But not with the real probabilities ...
- Define

$$q_s = \pi_s(1 + r)$$

## Risk-neutral Probabilities

If the market is free of arbitrage,

$$q_s > 0, \qquad \sum_{s=1}^{S} q_s = 1$$

form a set of probabilites, the so-called risk-neutral probabilities.

# From State Pricing to Risk-neutral Pricing

- If the market is free of arbitrage, prices can be expressed as

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \pi_s = \sum_{s=1}^{S} \mathrm{CF}_s \frac{q_s}{1+r}$$

- Consequently, prices can be expressed as discounted expected cash-flows!

## Risk-neutral Pricing

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \frac{q_s}{1+r} = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[\mathrm{CF}]$$

- This is a fundamental insight that holds in much more general markets!
- **1st Warning:** The risk-neutral probabilities are different from the real physical probabilities.
- **2nd Warning:** Investors are not risk-neutral, but prices are formed as if they were risk-neutral (but under **different probabilities**).

# Example: Risk-neutral Pricing

# Summary: State Pricing in a Complete Market

- State prices $\pi_s$ are the prices for a security paying one dollar in state $s$ and zero else.
- They can be determined by solving the linear system $X^\top \pi = p$, which has a unique solution if and only if the market is complete.
- The market is free of arbitrage if and only if $\pi_s > 0$ for all states $s$. If the market is arbitrage-free, the risk-neutral probabilities exists and are compounded state prices, i.e., $q_s = \pi_s(1+r)$.
- Given a vector of state prices $\pi$ or risk-neutral probabilities $q$, the price of an asset with cash-flow vector CF is given by

$$P_0 = \sum_{s=1}^{S} \mathrm{CF}_s \pi_s = \sum_{s=1}^{S} \mathrm{CF}_s \frac{q_s}{1+r}.$$

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# Example: Three States

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# Structure of One Period

$$
\begin{pmatrix} S_0 \\ B_0 \end{pmatrix}
\begin{array}{l}
\xrightarrow{p_u} \begin{pmatrix} S_0 \cdot (1+u) \\ B_0 \cdot (1+r) \end{pmatrix} \\[2em]
\xrightarrow[p_d]{} \begin{pmatrix} S_0 \cdot (1+d) \\ B_0 \cdot (1+r) \end{pmatrix}
\end{array}
$$

- Set $B_0 = 1$. Then:

$$
X = \begin{pmatrix} S_0 \cdot (1+u) & 1+r \\ S_0 \cdot (1+d) & 1+r \end{pmatrix}, \qquad p = \begin{pmatrix} S_0 \\ 1 \end{pmatrix}, \qquad u > r > d
$$

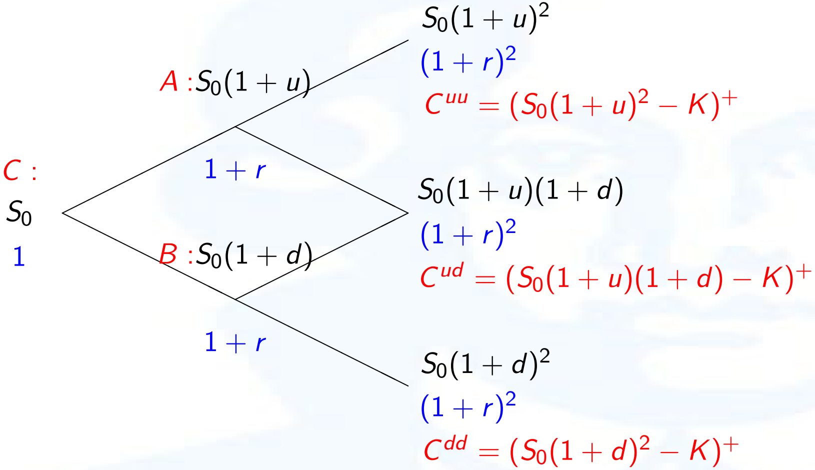
- Determine the price $C_0$ of a derivative with payoff $C = \begin{pmatrix} C^u \\ C^d \end{pmatrix}$.
- One obtains

$$
C_0 = \frac{1}{1+r} \cdot \Big[ C^u \underbrace{\frac{r-d}{u-d}}_{=q_u} + C^d \underbrace{\frac{u-r}{u-d}}_{=q_d} \Big]
$$

# Binomial Trees

- If we want to price financial derivatives, the one-period state pricing model is too simplistic.
- We extend the idea from the one-period model to a binomial tree.
- We consider trees with one stock $S$ and one risk-free asset $B$.
- The risk-free rate is exogeneously given and denoted by $r$.
- In each period, the stock can either increase by $u$ or decrease by $d$.
- We assume $u > r > d$. This condition ensures that the market is free-of arbitrage.
- The risk-neutral probability for an up-state is given by $q = q_u = \frac{r-d}{u-d}$.

# Two-Period Model



# Two-Period Model

- Node A:

$$C^u = \frac{1}{1+r}\left[qC^{uu} + (1-q)C^{ud}\right]$$

- Node B:

$$C^d = \frac{1}{1+r}\left[qC^{ud} + (1-q)C^{dd}\right]$$

- Node C:

$$\begin{aligned} C_0 &= \frac{1}{1+r}\left[qC^u + (1-q)C^d\right] \\ \Rightarrow C_0 &= \frac{1}{(1+r)^2}\left[q^2C^{uu} + 2q(1-q)C^{ud} + (1-q)^2C^{dd}\right] \end{aligned}$$

# Multi-Period Model: Binomial Coefficients

# Multi-Period Model

- Extending this idea to an arbitrary number of periods leads to the following closed-form solution

$$C_0 = \frac{1}{(1+r)^T} \sum_{i=0}^{T} \binom{T}{i} q^i (1-q)^{T-i} C_T^{(i)}$$

  where $\binom{T}{i} = \frac{T!}{i!(T-i)!}$ denotes the binomial coefficient. It counts the number of paths leading to node $i$. $\rightarrow$ Pascal's triangle
- For a call option the terminal payoff is given by

$$C_T^{(i)} = \left(S_0 (1+u)^i (1+d)^{T-i} - K\right)^+$$

- For a put option the terminal payoff is given by

$$C_T^{(i)} = \left(K - S_0 (1+u)^i (1+d)^{T-i}\right)^+$$

# Example: Multi-Period Model

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# Black-Scholes-Model

- Mathematically involved model in continuous time; stock price:

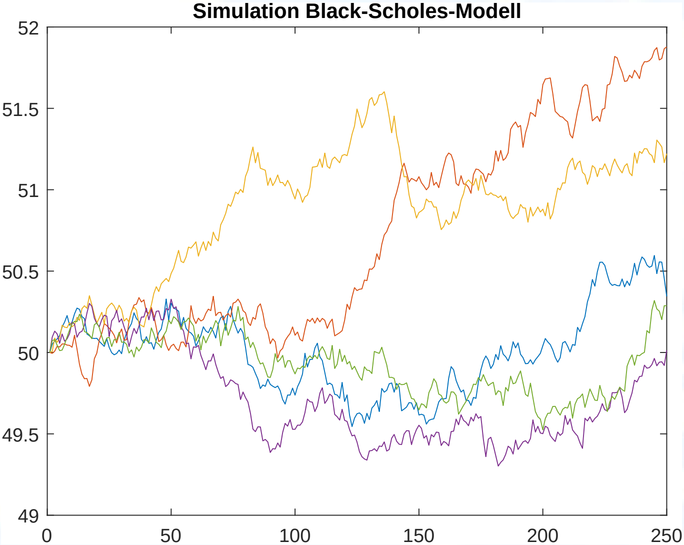
$$dS_t = S_t \mu \mathrm{d}t + S_t \sigma \mathrm{d}W_t$$

- $W_t$ Wiener process; normally distributed returns
- Interpretation for "small" time step $\Delta t$:

$$S_{t+\Delta t} - S_t = S_t \mu \Delta t + S_t \sigma \underbrace{\left(W_{t+\Delta t} - W_t\right)}_{\sim \mathcal{N}(0,\Delta t)}$$

- Idea: Start with a binomial model and increase the number of time steps in $[0, T]$.
- In the limit the binomial model converges to the continuous Black-Scholes model.

# Simulation of the Black-Scholes Model



# Black-Scholes Formula

- A call (put) option with maturity at time $T$ has the terminal payoff

$$C_T = \max\{S_T - K; 0\}, \qquad P_T = \max\{K - S_T; 0\}$$

- In the more complicated Black-Scholes setting, its current price is given by

$$C_0 = \mathbb{E}^{\mathbb{Q}}[C_T]e^{-rT}$$

- After some painful calculations, one obtains

## Black-Scholes Formula

$$C_0 = S_0\Phi(d_1) - K \cdot \mathrm{e}^{-rT}\Phi(d_2), \qquad P_0 = C_0 - S_0 + K \cdot \mathrm{e}^{-rT}$$

with

$$d_1 = \frac{\ln(S_0/K) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

# Example: Black-Scholes Formula

# Application: Convertible Bond

- Structure of a convertible bond
  - Fixed coupon payments at a rate $c$ until maturity.
  - At maturity, the *buyer* of the convertible bond has the right (but not the obligation) to reclaim the notional $N$ or to claim a number $k$ of stocks of the emitting company, i.e., $k \cdot S_T$.
- The buyer will claim the stocks if the stocks are worth more than the notional, $k \cdot S_T > N$, i.e., if $k \cdot S_T - N > 0$.
  $\Rightarrow$ Buyer holds a call option!
- Payoff structure:

| time | 1 | 2 | ... | $T$ |
|---|---|---|---|---|
| Cash flow | $c$ | $c$ | ... | $c + N + \max\{k \cdot S_T - N, 0\}$ |

  with $k = N/S_0$
- Structure: guaranteed coupon payments + long call option.

# Application: Reverse Convertible Bond

- Structure of a convertible bond
  - Fixed coupon payments at a rate $c$ until maturity.
  - At maturity, the *emitter* of the reverse convertible bond has the right (but not the obligation) to pay back the notional $N$ or to deliver a number $k$ of stocks, i.e., $k \cdot S_T$.
- The emitter will deliver the stocks if the stocks are worth less than the notional, $k \cdot S_T < N$, i.e., if $N - k \cdot S_T > 0$.
  $\Rightarrow$ Buyer is short in a put option!
- Payoff structure:

| time | 1 | 2 | ... | $T$ |
|---|---|---|---|---|
| Cash flow | $c$ | $c$ | ... | $c + N - \max\{N - k \cdot S_T, 0\}$ |

  with $k = N/S_0$
- Structure: guaranteed coupon payments – put option.

# Critique: Black-Scholes Model

- Volatility, interest rate, expected return are assumed to be constant. $\longrightarrow$ Volatility Smile
- Returns are assumed to be normally distributed. $\longrightarrow$ Underestimation of extreme events.
- Model builds upon a complete market without frictions (no taxes, transaction costs, short-selling constraints, . . . ).
- Implied volatility $\neq$ historical volatility
  - These caveats become visible if one investigates what volatilities are necessary to explain option prices by the Black-Scholes formula.
  - Implied volatility is not constant, but depends on $K$ and $T$.
  - If the option is at-the-money, implied volatility is lowest (volatility smile).

# Volatility Smile